

CURRENT-MODE MODELING FOR PEAK, VALLEY AND EMULATED CONTROL METHODS

Reference Guide for Fixed-Frequency, Continuous Conduction-Mode Operation

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Current-Mode Control

For current-mode control there are three things to consider:

1. Current-mode operation. An ideal current-mode converter is only dependent on the dc or average inductor current. The inner current loop turns the inductor into a voltage-controlled current source, effectively removing the inductor from the outer voltage control loop at dc and low frequency.
2. Modulator gain. The modulator gain is dependent on the effective slope of the ramp presented to the modulating comparator input. Each operating mode will have a unique characteristic equation for the modulator gain.
3. Slope compensation. The requirement for slope compensation is dependent on the relationship of the average current to the value of current at the time when the sample is taken. For fixed-frequency operation, if the sampled current were equal to the average current, there would be no requirement for slope compensation.

Current-Mode Operation

Whether the current-mode converter is peak, valley, average, or sample-and-hold is secondary to the operation of the current loop. As long as the dc current is sampled, current-mode operation is maintained. The current-loop gain splits the complex-conjugate pole of the output filter into two real poles, so that the characteristic of the output filter is set by the capacitor and load resistor. Only when the impedance of the output inductor equals the current-loop gain does the inductor pole reappear at higher frequencies.

To understand how this works, voltage-mode operation is examined. The basic concept of pulse-width modulation is used to establish the criteria for the modulator gain. This allows a linear model to be developed, illustrating the dc- and ac-gain characteristics.

Having established the basic modulator concept, the current loop is added by sensing the inductor current, and feeding the sensed signal back to the modulator.

Modulator Gain

For simplicity, the buck regulator is used to illustrate the operation.

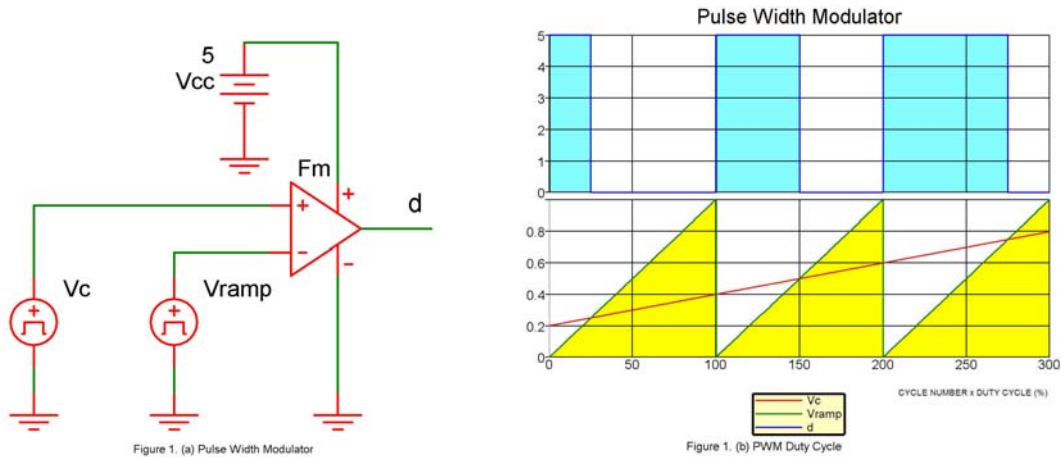


Figure 1. Pulse-width modulator

Voltage-Mode

A comparator is used to modulate the duty cycle. Fixed-frequency operation is shown in Figure 1, where a sawtooth voltage ramp is presented to the inverting input. The control or error voltage is applied to the non-inverting input. The modulator gain F_m is defined as the change in control voltage which causes the duty cycle to go from 0% to 100%:

$$F_m = \frac{d}{v_C} = \frac{1}{V_{RAMP}}$$

The modulator voltage gain K_m , which is the gain from the control voltage to the switch voltage is defined as:

$$K_m = V_{IN} \cdot F_m = \frac{V_{IN}}{V_{RAMP}}$$

For voltage-mode operation, the control-to-output transfer function is found by multiplying the modulator voltage gain by the output filter response. With $V_{IN} = 10V$ and $V_{RAMP} = 1V$, $K_m = 10$ which is 20dB. Figure 2 shows the schematic, linear model and frequency response plot. The complex-conjugate pole of the LC output filter is clearly seen, with the resulting 180° phase shift.

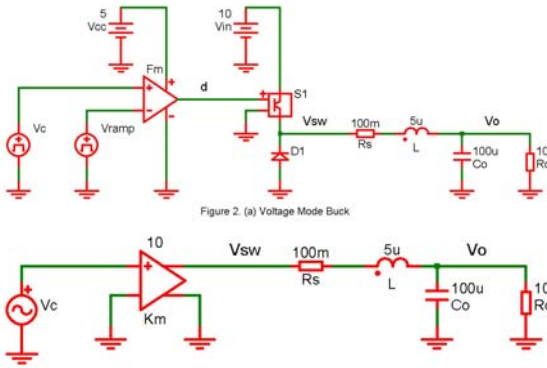


Figure 2. (a) Voltage Mode Buck

Figure 2. (b) Voltage Mode Buck - Linear Model

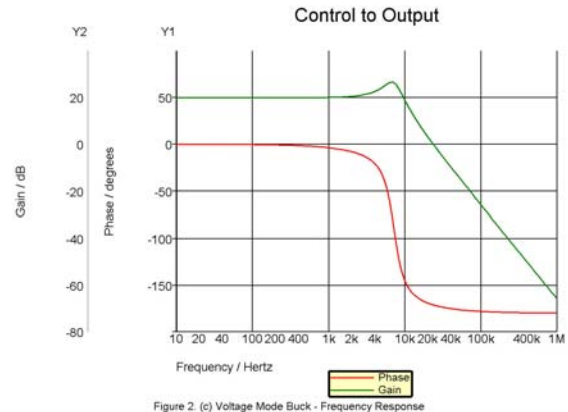


Figure 2. (c) Voltage Mode Buck - Frequency Response

Figure 2. (a) Voltage-mode buck, (b) Linear model, (c) Frequency response

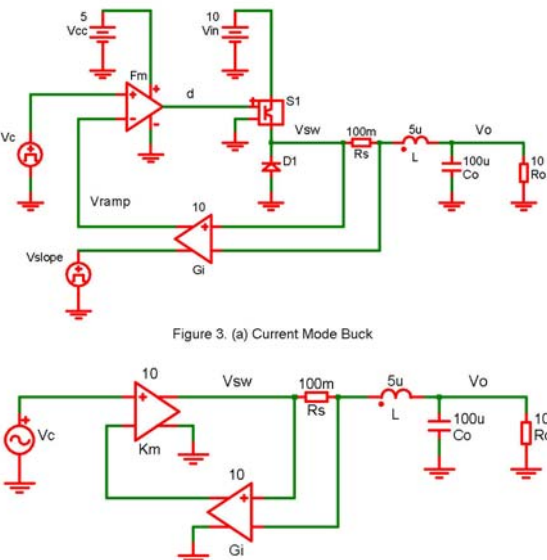


Figure 3. (a) Current Mode Buck

Figure 3. (b) Current Mode Buck - Linear Model

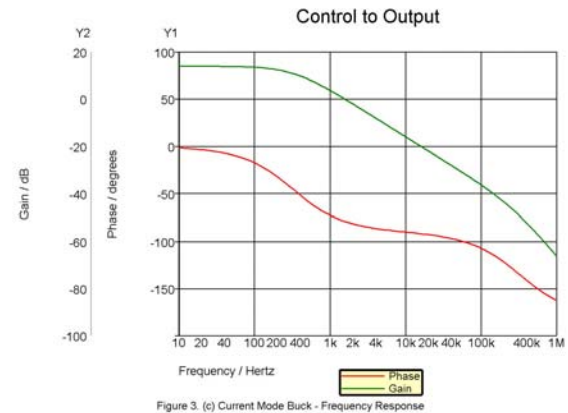


Figure 3. (c) Current Mode Buck - Frequency Response

Figure 3. (a) Current-mode buck, (b) Linear model, (c) Frequency response

Current-Mode

The same PWM function occurs for current-mode control, except that the ramp is created by monitoring the inductor current. This signal is comprised of two parts: the ac ripple current, and the dc or average value of the inductor current. The output of the current-sense amplifier G_i is summed with an external ramp V_{SLOPE} , to produce V_{RAMP} at the inverting input of the comparator.

In Figure 3 the effective $V_{RAMP} = 1V$, which was used for the voltage-mode modulator. With $V_{IN} = 10V$, the modulator voltage gain $K_m = 10$.

The linear model for the current loop is an amplifier which feeds back the dc value of the inductor current, creating a voltage-controlled current source. This is what makes the inductor disappear at dc and low frequency. The ac ripple current sets the modulator gain.

The current-sense gain is usually expressed as the product of the current-sense amplifier gain and the sense resistor:

$$R_i = G_i \cdot R_s$$

The current-sense gain is an equivalent resistance, the units of which are volts/amp. The current-loop gain is the product of the modulator voltage gain and the current-sense gain, which is also in volts/amp. The modulator voltage gain is reduced by the equivalent divider ratio of the load resistor R_O and the current-loop gain $K_m \cdot R_i$. This sets the dc value of the control-to-output gain. Neglecting the dc loss of the sense resistor:

$$\frac{V_O}{V_C} = K_m \cdot \frac{R_O}{R_O + K_m \cdot R_i}$$

This is usually written in factored form:

$$\frac{V_O}{V_C} = \frac{R_O}{R_i} \cdot \frac{1}{1 + \frac{R_O}{K_m \cdot R_i}}$$

The dominant pole in the transfer function appears when the impedance of the output capacitor equals the parallel impedance of the load resistor and the current-loop gain:

$$\omega_p = \frac{1}{C_O} \cdot \left(\frac{1}{R_O} + \frac{1}{K_m \cdot R_i} \right)$$

The inductor pole appears when the impedance of the inductor equals the current-loop gain:

$$\omega_L = \frac{K_m \cdot R_i}{L}$$

The current loop creates the effect of a lossless damping resistor, splitting the complex-conjugate pole of the output filter into two real poles.

For current-mode control, the ideal steady-state modulator gain may be modified depending upon whether the external ramp is fixed, or proportional to some combination of input and output voltage. Further modification of the gain is realized when the input and output voltages are perturbed to derive the effective small-signal terms. However, the concepts remain valid, despite small-signal modification of the ideal steady-state value.

Slope Compensation

The difference between the average inductor current and the dc value of the sampled inductor current can cause instability for certain operating conditions. This instability is known as sub-harmonic oscillation, which occurs when the inductor ripple current does not return to its initial value by the start of next switching cycle. Sub-harmonic oscillation is normally characterized by observing alternating wide and narrow pulses at the switch node. Adding an external ramp (slope compensation) to the current-sense signal prevents this oscillation.

Formal derivation of the criteria for slope compensation is covered in reference [1]. For the purpose of this analysis, a discussion of feed-forward techniques and some illustrations will suffice.

For the buck regulator, the modulator voltage gain K_m was found to be V_{IN} / V_{RAMP} . For voltage-mode operation, the gain varies with V_{IN} . Feed-forward techniques are often employed to stabilize the gain. This is typically done by generating V_{RAMP} with a voltage-controlled current source or fixed resistor charging a capacitor from V_{IN} .

Peak Current-Mode

Peak current-mode control is often referred to as having inherent line feed-forward. This is basically true, but is not quite ideal. The sensed inductor up-slope, which is used as V_{RAMP} / T for the modulator is equal to $(V_{IN} - V_O) \cdot R_i / L$. In order to stabilize the gain, an external ramp of $V_{SLOPE} / T = V_O \cdot R_i / L$ must be added to the current-sense signal. The resultant $V_{RAMP} / T = V_{IN} \cdot R_i / L$.

Figure 4 shows the under-damped condition, where sub-harmonic oscillation occurs with a duty cycle greater than 50%. The relationship of Q as shown in the graphs is covered in the section on sampling gain. To demonstrate the under-damped condition, $V_{SLOPE} / T = 0.1 \cdot V_O \cdot R_i / L$.

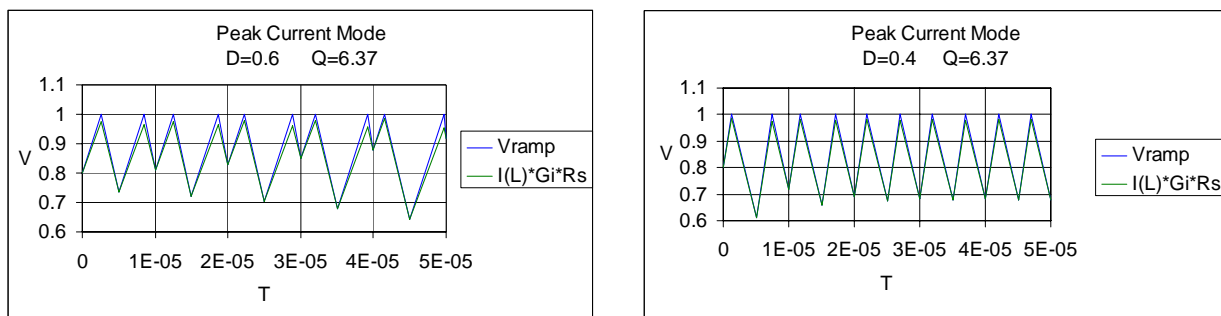


Figure 4. Peak current-mode sub-harmonic oscillation. For $D < 0.5$, sub-harmonic oscillation is damped. For $D > 0.5$, sub-harmonic oscillation builds with insufficient slope compensation.

By adding a compensating ramp equal to the down-slope of the inductor current, any tendency toward sub-harmonic oscillation is damped within one switching cycle. This is demonstrated graphically in Figure 5.

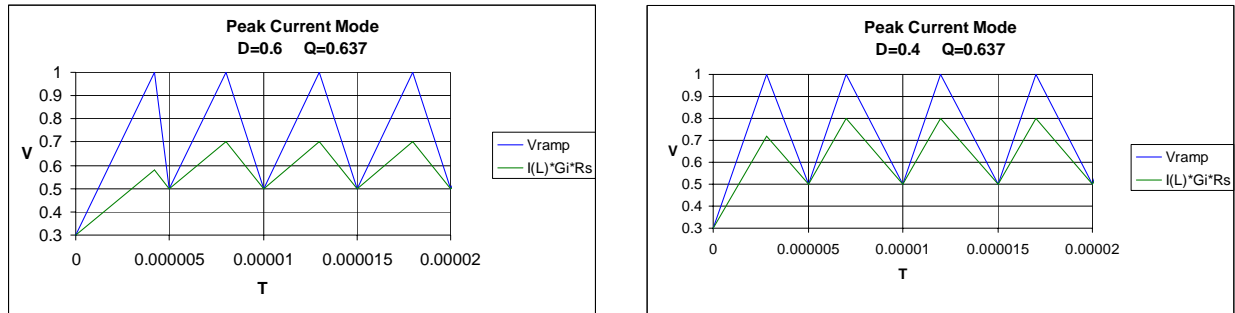


Figure 5. Optimally compensated peak current-mode buck.

For peak current-mode control, when the compensating ramp is equal to one-half the down-slope of the inductor current, infinite line rejection is achieved. Though a desirable operating point, this represents a special case. As the theoretical limit for stability of the current loop, the tendency toward sub-harmonic oscillation increases as the duty cycle approaches unity. To ensure stability of the current loop, the optimal compensating slope remains equal to one times the down-slope of the inductor current.

Valley Current-Mode

For valley current-mode, the down-slope of the inductor current is presented to the modulator, which is $V_O \cdot R_i / L$. This transposes the function of the external ramp. It is now necessary to use slope compensation equal to the up-slope of the inductor current, so $V_{SLOPE} / T = (V_{IN} - V_O) \cdot R_i / L$. Again, the resultant $V_{RAMP} / T = V_{IN} \cdot R_i / L$.

Figure 6 shows the under-damped condition, where sub-harmonic oscillation occurs with a duty cycle less than 50%. To demonstrate the under-damped condition, $V_{SLOPE} / T = 0.1 \cdot (V_{IN} - V_O) \cdot R_i / L$.

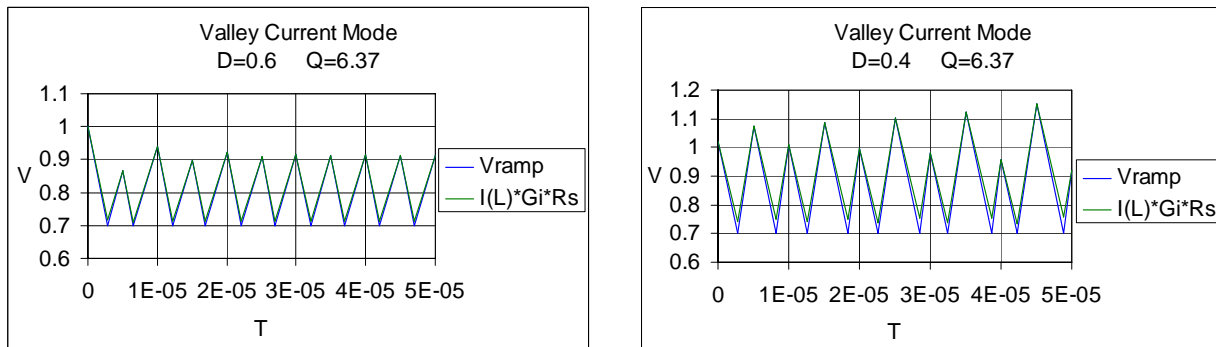


Figure 6. Valley current-mode sub-harmonic oscillation. For $D > 0.5$, sub-harmonic oscillation is damped. For $D < 0.5$, sub-harmonic oscillation builds with insufficient slope compensation.

Emulated Peak Current-Mode

For emulated peak current-mode, the valley current is sampled on the down-slope of the inductor current. This is used as the dc value of current to start the next cycle. A slope-compensating ramp is added to produce V_{RAMP} at the modulator input.

The primary application for emulated peak current-mode is high input voltage to low output voltage operating at a narrow duty cycle. In any practical design, device capacitance and wiring inductance may cause a significant leading-edge spike on the current-sense waveform, followed by an extended period of ringing. By sampling the inductor current at the end of the switching cycle and adding an external ramp, the minimum on-time can be significantly reduced, without the need for blanking or filtering which is normally required for peak current-mode control.

To determine the correct slope compensation, the most salient feature is the absence of any ramp from the inductor, since only the dc value of the valley current is sampled. Formal derivation in reference [1] has shown the optimal compensation to be $V_{SLOPE} / T = V_{RAMP} / T = V_{IN} \cdot R_i / L$. This is consistent with the results for both peak and valley buck regulators.

Figure 7 shows the under-damped condition, with $V_{SLOPE} / T = 0.55 \cdot V_{IN} \cdot R_i / L$.

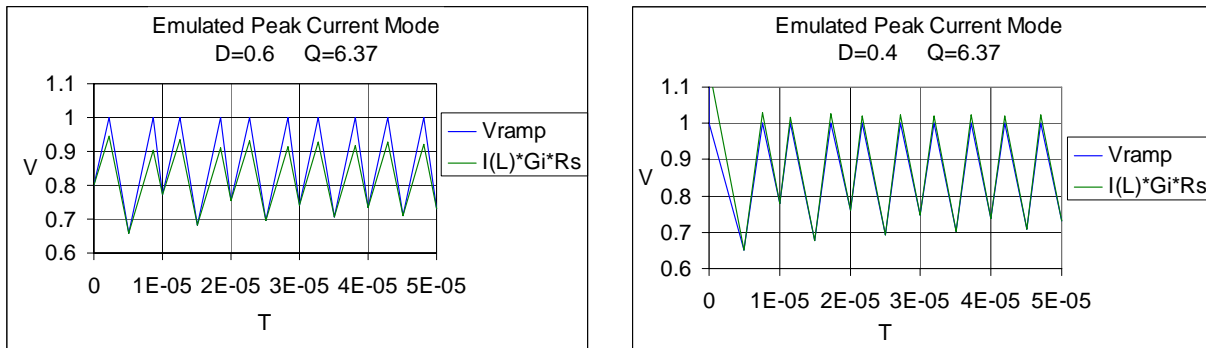


Figure 7. Emulated peak current-mode sub-harmonic oscillation. Tendency for sub-harmonic oscillation is independent of duty cycle. Even with minimal damping, it will eventually die out.

Since the slope compensation requirement is independent of duty cycle, an interesting observation can be made. If the slope of the ramp is made less than $0.5 \cdot V_{IN} \cdot R_i / L$, the circuit will exhibit sub-harmonic oscillation at any duty cycle.

General Slope Compensation Criteria

For any mode of operation (peak, valley or emulated), the optimal slope of the ramp presented to the modulating comparator input is equal to the sum of the absolute values of the inductor up-slope and down-slope scaled by the current-sense gain. This will cause any tendency toward sub-harmonic oscillation to damp in one switching cycle.

For the buck regulator, this is equivalent to a ramp whose slope is $V_{IN} \cdot R_i / L$.

$$\text{Up-slope} = (V_{IN} - V_O) \cdot R_i / L$$

$$\text{Down-slope} = V_O \cdot R_i / L$$

For the boost regulator, this is equivalent to a ramp whose slope is $V_O \cdot R_i / L$.

$$\text{Up-slope} = V_{IN} \cdot R_i / L$$

$$\text{Down-slope} = (V_O - V_{IN}) \cdot R_i / L$$

For the buck-boost regulator, this is equivalent to a ramp whose slope is $(V_{IN} + V_O) \cdot R_i / L$.

$$\text{Up-slope} = V_{IN} \cdot R_i / L$$

$$\text{Down-slope} = V_O \cdot R_i / L$$

To avoid confusion, V_{IN} and V_O represent the magnitude of the input and output voltages as a positive quantity. By identifying the appropriate sensed inductor slope, it is easy to find the correct slope-compensating ramp.

Sampling Gain

A current-mode switching regulator is a sampled-data system, the bandwidth of which is limited by the switching frequency. Beyond half the switching frequency, the response of the inductor current to a change in control voltage is not accurately reproduced. In order to quantify this effect for linear modeling, the continuous-time model of reference [2] successfully placed the sampling-gain term in the closed-current feedback loop. This allows accurate modeling of the control-to-output transfer function using the term $H_c(s)$.

In order to accurately model the current loop, the unified model of reference [3] placed the sampling-gain term in the forward path. For peak or valley current-mode with a fixed slope compensation ramp, this also accurately models the control-to-output transfer function using the term $F_m(s)$.

To develop the theory for emulated current-mode control, reference [1] used a fresh approach, deriving general gain parameters which are consistent with both models. In addition, a new representation of the sampling-gain term for the closed current-loop was developed, identifying limitations of the forward-path sampling-gain term.

Figure 8 (a) represents the unified form of the model, with K being the feed-forward term. In (b) K_n is the dc audio susceptibility coefficient from the continuous-time model. The linear model sampling-gain terms as shown in Figure 2 are defined as:

$$H_p(s) = \frac{1}{1 + s \cdot \frac{Q}{\omega_n}} \quad H(s) = 1 + s \cdot K_e + \frac{s^2}{\omega_n^2} \quad \omega_n = \frac{\pi}{T}$$

K_e is a new term which emerged from the derivation of the closed-loop expression for $H(s)$. This derivation used slope-compensation terms other than the classic fixed ramp for peak or valley current-mode. K_e could be expressed as $\frac{1}{\omega_n \cdot Q_e}$ but this seems rather pointless, as Q_e would need a value of infinity for $K_e = 0$. To date, no method has been found which successfully incorporates K_e into the open-loop expression for $H_p(s)$. Use of $H_p(s)$ is limited to peak or valley current-mode with a fixed slope-compensating ramp, for which the value of $K_e = 0$.

To place either sampling-gain term into the linear models:

$$F_m(s) = F_m \cdot H_p(s) \quad G_i(s) = G_i \cdot H(s)$$

The accuracy limit for the sampling-gain term is identified by comparing Q to the modulator voltage gain K_m and the feed-forward term K . Q is directly related to the slope-compensation requirement. The derivation starts with the ideal steady-state modulator gain. The physical reason being that at the switching frequency, the relative slopes are fixed with respect to the period T . A change in control voltage is then related to a change in average inductor current. Any transfer function which is solely dependent on K_m in the forward dc-gain path will have excellent

agreement to the switching model up to half the switching frequency. Any transfer function which includes K in the forward dc-gain path will show some deviation at half the switching frequency.

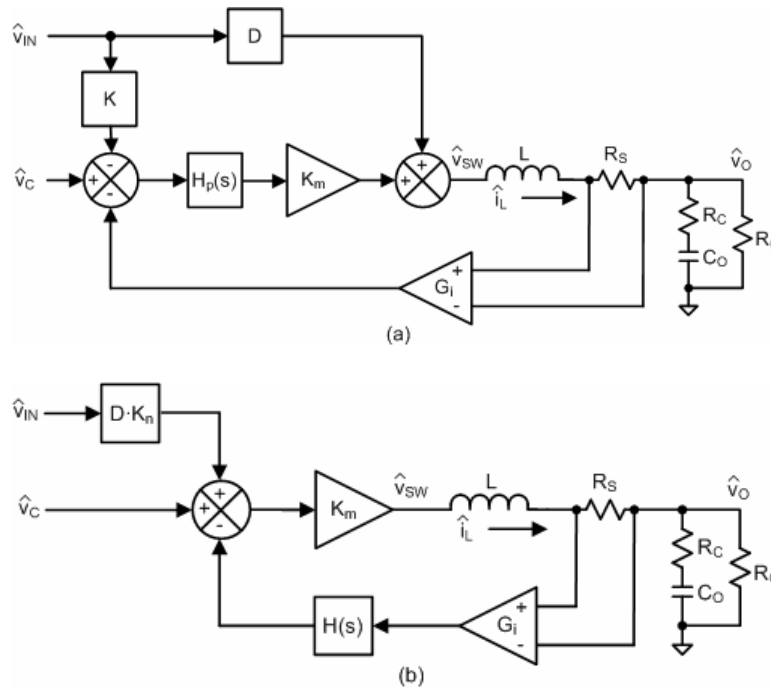


Figure 8. Buck regulator with sampling-gain terms. (a) Sampling gain $H_p(s)$ in the forward path. (b) Sampling gain $H(s)$ in the closed current-loop feedback path.

Simplified Transfer Functions

No assumptions for simplification were made during the derivation of the transfer functions. The only initial assumptions are the ones generally accepted to be valid for a first order analysis. Voltage sources, current sources and switches are ideal, with no delays in the control circuit. Amplifier inputs are high impedance, with no significant loading of the previous stage. Simplification of the results was made after the complete derivation, which included all terms. See the appendix for the complete derivation of all transfer functions.

In order to show the factored form, the simplified transfer functions assume poles which are well separated by the current-loop gain. Expressions for the low frequency model do not show the additional phase shift due to the sampling effect. The control-to-output transfer function with sampling-gain term accurately represents the circuit's behavior to half the switching frequency. The line-to-output expressions for audio susceptibility are accurate at dc, but diverge from the actual response as frequency increases.

The current-sense gain $R_i = G_i \cdot R_S$, where G_i is the current-sense amplifier and R_S is the sense resistor.

To include the sampling-gain term in the control-to-output transfer function, replace $1 + \frac{s}{\omega_L}$ with $1 + \frac{s}{\omega_n \cdot Q} + \frac{s^2}{\omega_n^2}$ in the low-frequency equations. This represents the closed current-loop sampling-gain term. Inclusion of this term in the line-to-output equations will not produce the same accuracy of results. For peak or valley current-mode with a fixed slope-compensating ramp, $\omega_n \cdot Q = \omega_L$.

Sampling Gain Q

Using a value of $Q = 0.637$ will cause any tendency toward sub-harmonic oscillation to damp in one switching cycle. With respect to the closed current-loop control-to-output function, the effective sampled-gain inductor pole is given by:

$$f_L(Q) = \frac{1}{4 \cdot T \cdot Q} \cdot \left(\sqrt{1 + 4 \cdot Q^2} - 1 \right)$$

This is the frequency at which a 45° phase shift occurs due to the sampling gain. For $Q = 0.637$, $f_L(Q)$ occurs at 24% of the switching frequency. For $Q = 1$, $f_L(Q)$ occurs at 31% of the switching frequency. For second-order systems, $Q = 1$ is normally associated with best transient response. $Q = 0.5$ ($\delta = 1$) is the criteria for critical damping. Using $Q = 1$ may make an incremental difference for the buck, but is inconsequential for the boost and buck-boost with the associated right-half-plane zero of ω_R . For the peak current-mode buck with a fixed slope-compensating ramp, the effective sampled-gain inductor pole is only fixed in frequency with respect to changes in line voltage when $Q = 0.637$. Proportional slope-compensation methods will achieve this for other operating modes.

To determine the effect of reducing the slope compensation in order to increase the voltage-loop bandwidth, an emulated peak current-mode buck with proportional slope-compensation (EPCM2) switching circuit was implemented in SIMPLIS. A standard type II 10MHz error amplifier was used for frequency compensation. With $T/L = 5\mu s/5\mu H$ and $R_i = 0.1V/A$, the best performance was achieved with $Q = 0.637$ for a crossover frequency of 40 kHz and 45° phase margin. By setting $Q = 1$ a crossover frequency of 50 kHz was achieved, again with 45° phase margin but reduced gain margin. This appears to be the practical limit for a stable voltage loop, at the expense of under-damping the current loop. With $Q = 1$, sub-harmonic oscillation is quite pronounced during transient response, but damps at steady state.

Simulation Programs

The choice of simulation program is important, since all SPICE programs do not calculate all parameters with the same degree of accuracy. For switching-model simulation, SIMPLIS is able to produce Bode plots directly from the switching model. This was used for the switching-model results.

The linear models were made with SIMetrix, which is the general purpose simulator for the SIMetrix/SIMPLIS program. This simulator only handles Laplace equations for s in numerical form, where the numerator order must be equal or less than the denominator order. PSpice is much better suited for linear models with Laplace functions in parameter form. It is more accurate than the SIMetrix/SIMPLIS program, but cannot produce Bode plots directly from the switching model. PSpice or a program with similar capability may be used to obtain the linear-model results.

Linear Models

The linear models shown in Figures 22, 23 and 24 represent the state of the art in simplicity, accuracy and ease of use. Verification of each linear model has been made to results from the switching model. In this manner, validation for any transfer function is possible, identifying the accuracy limit of the linear model.

Transfer Functions in Impedance Form

For all transfer functions:

$$Z_O = \left(\frac{1}{s \cdot C_O} + R_C \right) \parallel R_O = \frac{R_O \cdot (1 + s \cdot C_O \cdot R_C)}{1 + s \cdot C_O \cdot (R_O + R_C)} \quad Z_L = s \cdot L + R_L + R_S$$

R_O represents the load resistance, while R represents the dc operating point V_O / I_O .

For a resistive load $R_O = R$.

For a non-linear load such as an LED, $R_O = R_D$, where R_D represents the dynamic resistance of the load at the operating point, plus any series resistance.

For a constant-current load, $R_O = \infty$.

G_V represents the error amplifier gain as a positive quantity. To find the open-loop transfer functions, let $G_V = 0$.

To include either sampling-gain term multiply K_m by $H_P(s)$, or R_i by $H(s)$.

Buck Regulator Example

For the peak current-mode buck example, comparisons of results from the switching circuit of Figure 10 were made to the linear model of Figure 22 using the sampling-gain term $H_p(s)$. In order to use the forward-path sampling-gain term, slope compensation was implemented with a fixed ramp. The results will be slightly different if a proportional ramp is used, as this modifies the modulator gain term K_m and feed-forward term K .

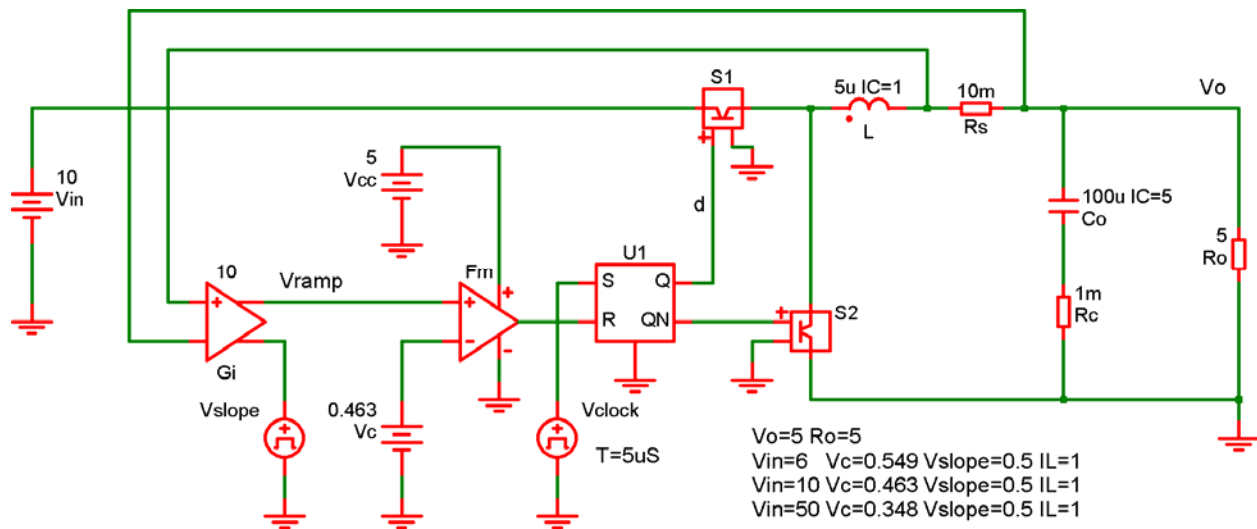


Figure 10. Buck switching model

Results for the control-to-output gain in Figure 11 show excellent agreement up to half the switching frequency. The line-to-output results in Figure 12 show some deviation due to the feed-forward term K being in the forward-gain path. Slope compensation was set for $Q = 0.637$. This can be seen from the current-loop crossover in Figure 13, which converges for all line voltages at 60 kHz or 30% of the switching frequency.

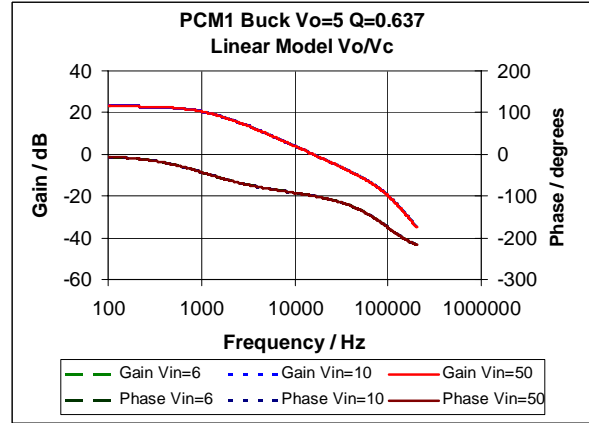
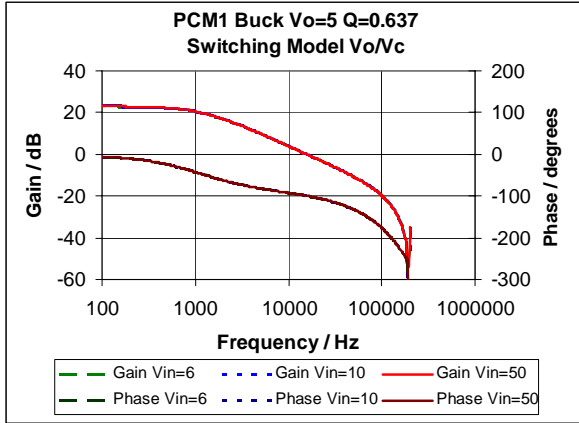


Figure 11. Buck control-to-output

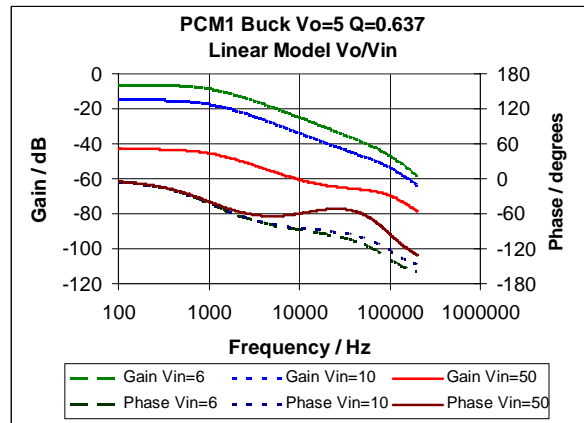
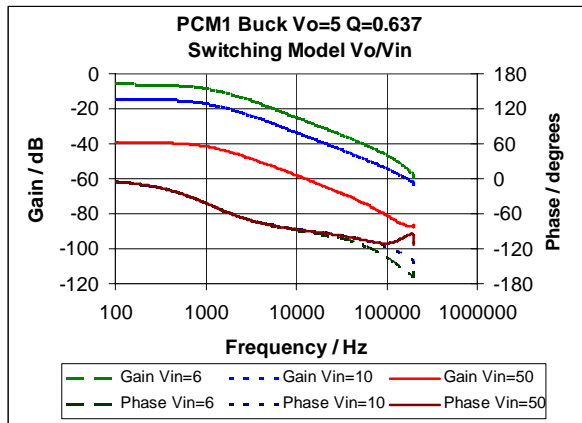


Figure 12. Buck line-to-output

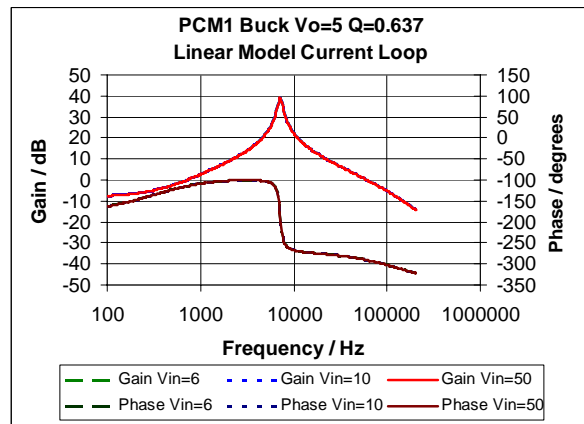
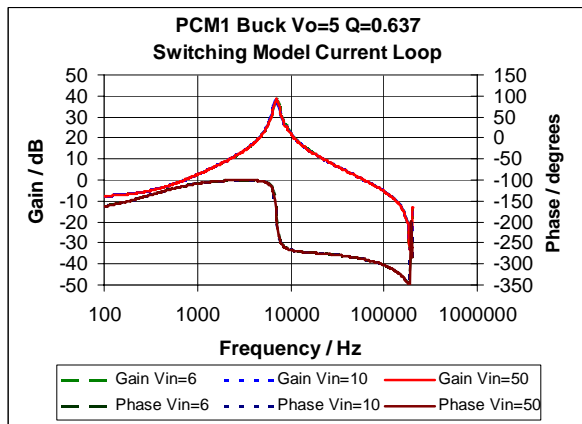


Figure 13. Buck current loop

Boost Regulator Example

For the peak current-mode boost example, comparisons of results from the switching circuit of Figure 14 were made to the linear model of Figure 23 using the sampling-gain term $H_p(s)$. In order to use the forward-path sampling-gain term, slope compensation was implemented with a fixed ramp. The results will be slightly different if a proportional ramp is used, as this modifies the modulator gain term K_m and feed-forward term K . For an actual boost converter implementation with a fixed ramp, it is only possible to get the optimal Q at one input voltage.

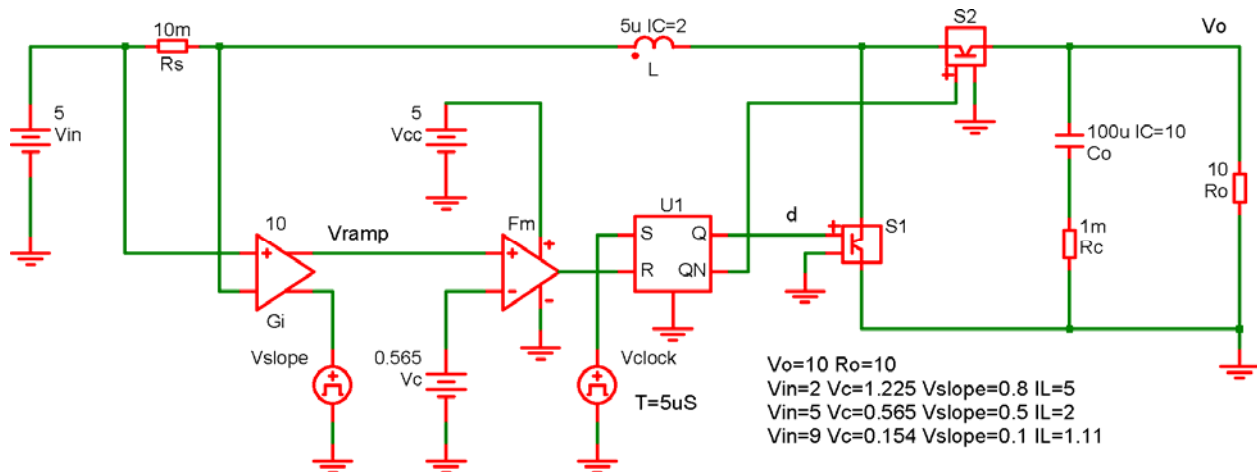


Figure 14. Boost switching model

Results for the control-to-output gain in Figure 15 show a slight deviation at half the switching frequency. The line-to-output results in Figure 16 show excellent agreement up to half the switching frequency. Slope compensation was set for $Q = 0.637$. This can be seen from the current-loop crossover in Figure 17, which converges for all line voltages at 60 kHz or 30% of the switching frequency.

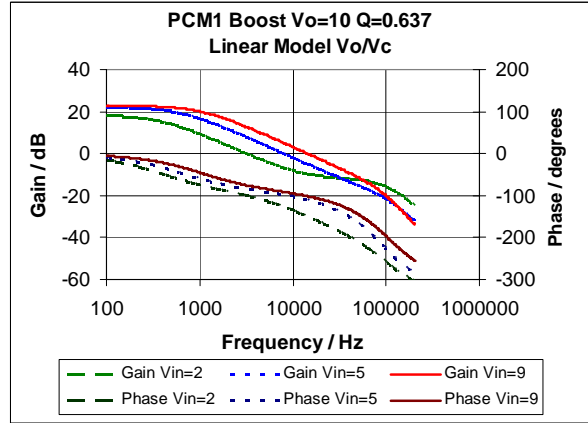
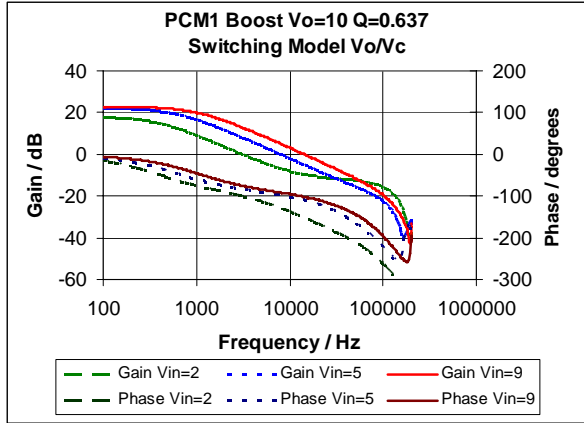


Figure 15. Boost control-to-output

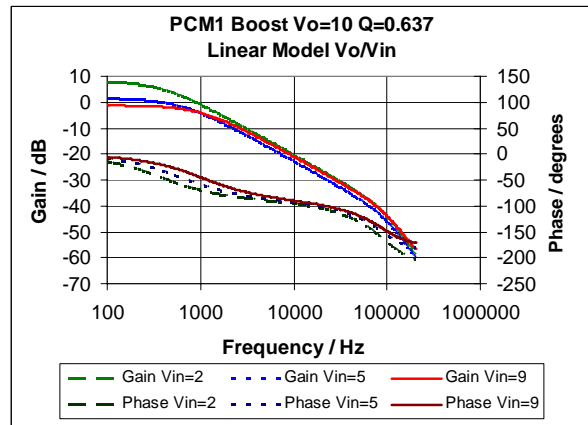
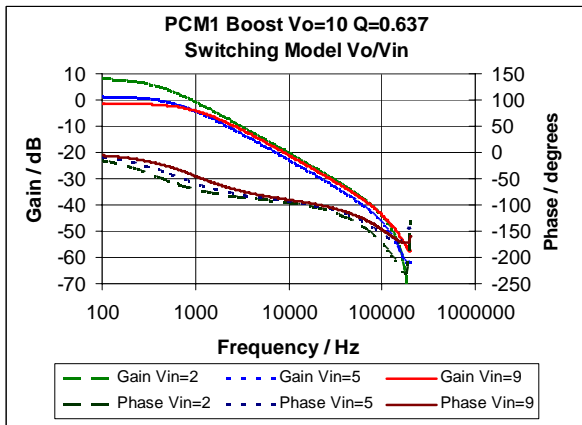


Figure 16. Boost line-to-output

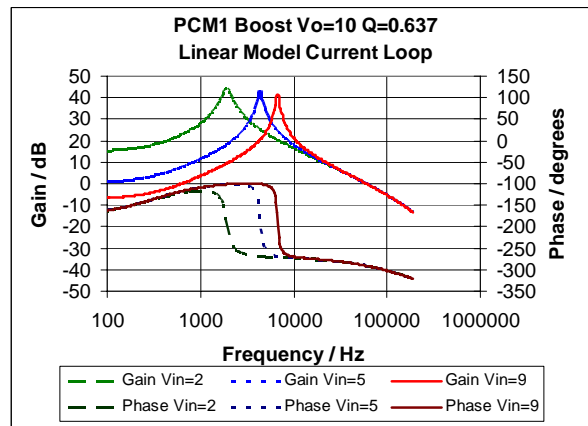
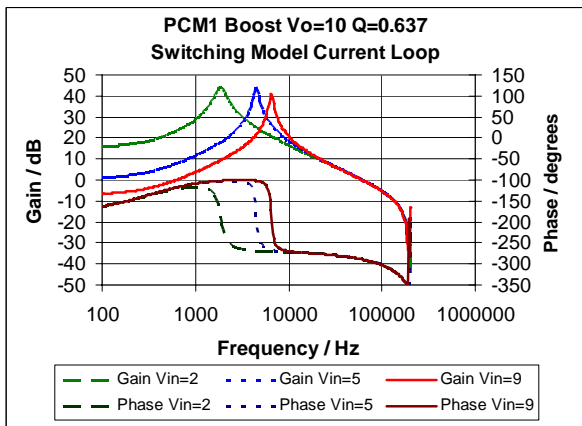


Figure 17. Boost current loop

Buck-Boost Regulator Example

For the peak current-mode buck-boost example, comparisons of results from the switching circuit of Figure 18 were made to the linear model of Figure 24 using the sampling-gain term $H_p(s)$. In order to use the forward-path sampling-gain term, slope compensation was implemented with a fixed ramp. The results will be slightly different if a proportional ramp is used, as this modifies the modulator gain term K_m and feed-forward term K .

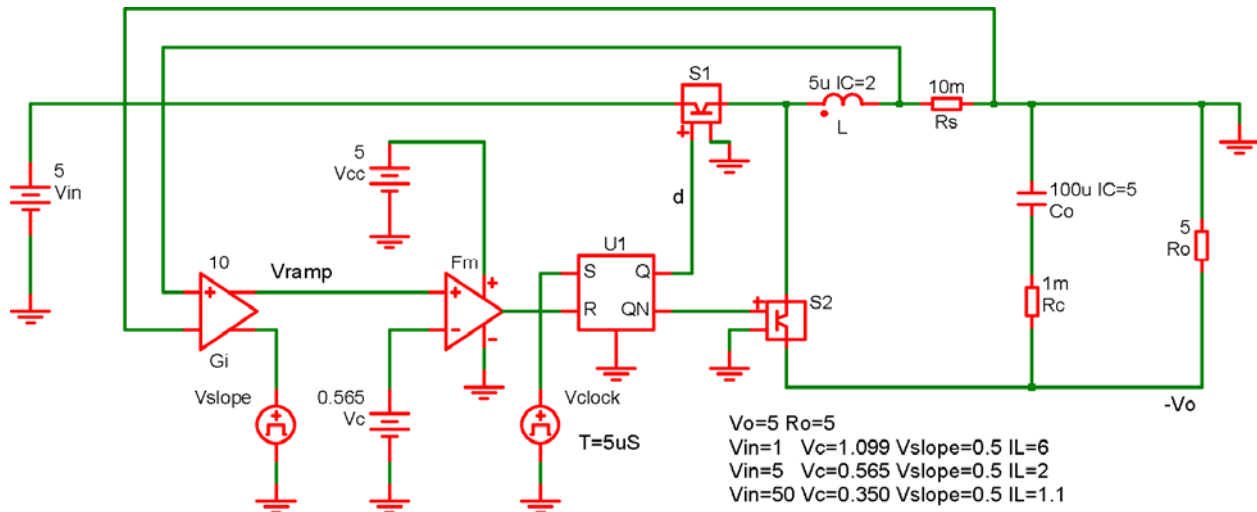


Figure 18. Buck-boost switching model

Results for the control-to-output gain in Figure 19 show the same slight deviation as the boost at half the switching frequency. Similar to the buck, the line-to-output results in Figure 20 show a more pronounced deviation due to K being in the forward-gain path. Slope compensation was set for $Q = 0.637$. This can be seen from the current-loop crossover in Figure 21, which converges for all line voltages at 60 kHz or 30% of the switching frequency.

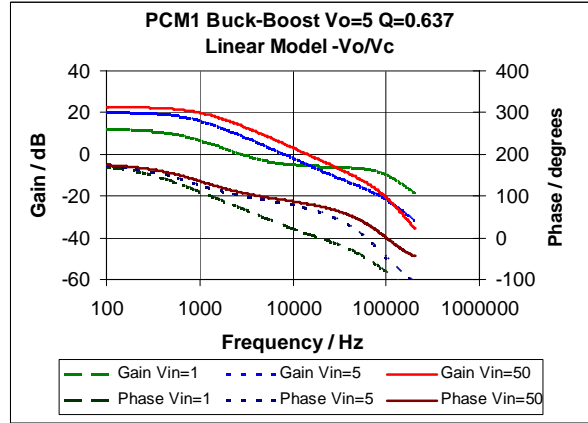
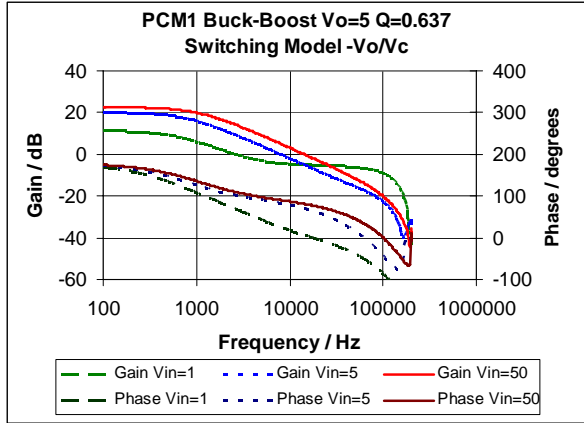


Figure 19. Buck-boost control-to-output

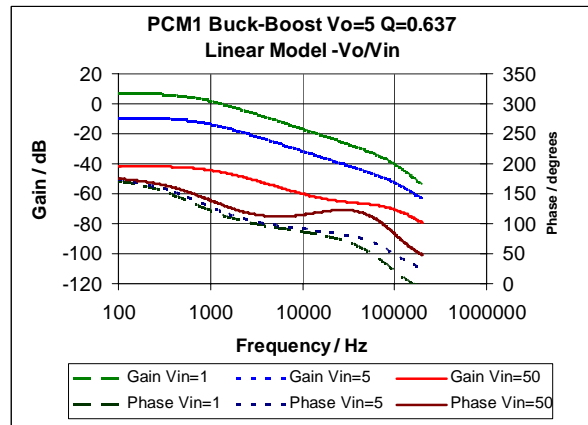
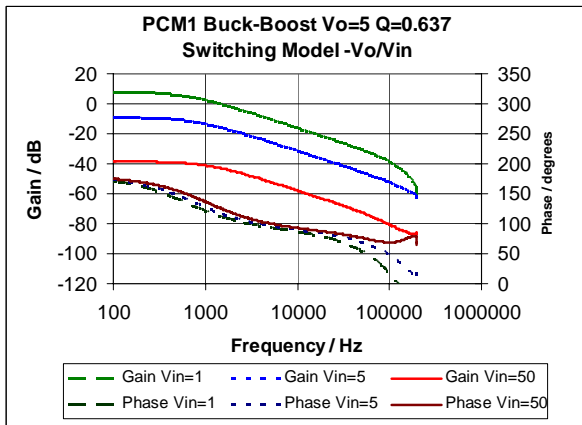


Figure 20. Buck-boost line-to-output

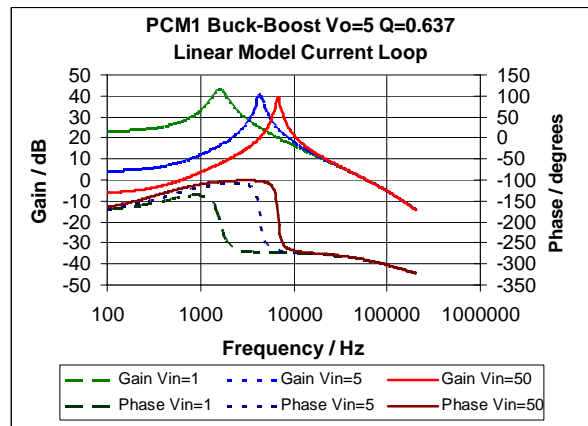
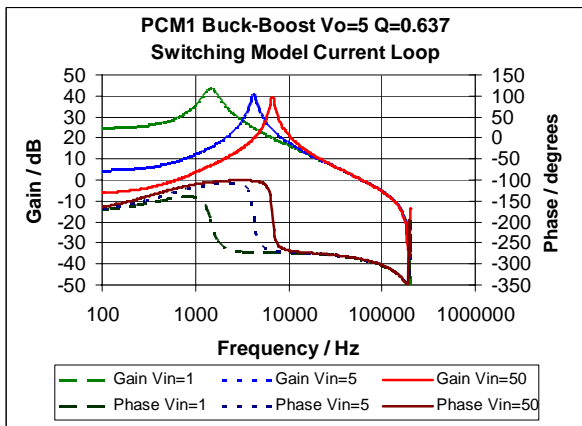


Figure 21. Buck-boost current loop

Current-Mode Buck – Transfer Functions in Impedance Form

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{1}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right)}$$

Line-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{D \cdot \left(\frac{1}{K_m} - \frac{K}{D}\right)}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right) + G_v}$$

Output Impedance:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right) + G_v}$$

Input Impedance:

$$\frac{\hat{v}_{IN}}{\hat{i}_{IN}} = \frac{R}{D^2} \cdot \frac{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right) + G_v}{\frac{K}{D} \cdot \left(1 + \frac{R + Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right) + G_v - \frac{1}{K_m} \cdot \left(\frac{R}{Z_O}\right)}$$

Control-to-Inductor Current:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right)}$$

Current Loop ($\hat{v}_C = 0$):

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\frac{1}{Z_O}\right)}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right)}$$

Current-Mode Boost

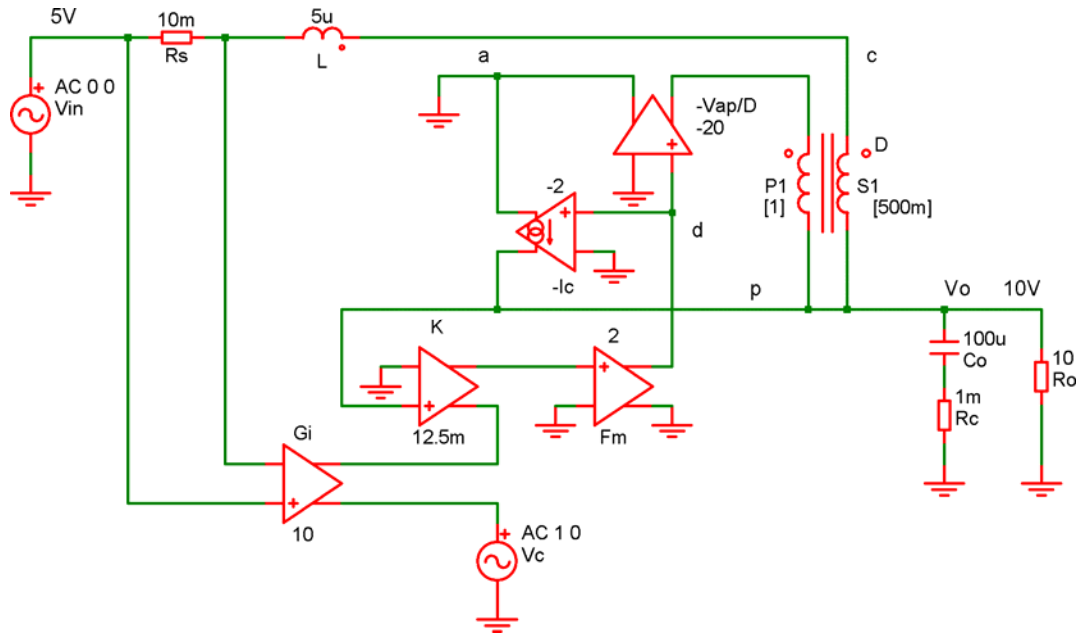


Figure 23. Boost linear model

Linear Model Coefficients

$$V_{ap} = V_O \quad D = \frac{V_O - V_{IN}}{V_O} \quad D' = 1 - D = \frac{V_{IN}}{V_O}$$

$$M = \frac{1}{D'} \quad R = \frac{V_O}{I_O} \quad I_C = \frac{V_{ap} \cdot M}{R} \quad F_m = \frac{K_m}{V_{ap}}$$

Simplified Transfer Functions

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i \cdot K_D} \cdot \frac{\left(1 - \frac{s}{\omega_R}\right) \cdot \left(1 + \frac{s}{\omega_Z}\right)}{\left(1 + \frac{s}{\omega_P}\right) \cdot \left(1 + \frac{s}{\omega_L}\right)}$$

Line-to-Output ($\hat{v}_C = 0$):

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{R_O \cdot D' \cdot K_n}{R_i \cdot K_D} \cdot \frac{1 + \frac{s}{\omega_Z}}{\left(1 + \frac{s}{\omega_P}\right) \cdot \left(1 + \frac{s}{\omega_L}\right)}$$

Where:

$$K_D = 1 + \frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) \quad K_n = \frac{1}{K_m} + \frac{R_i}{R \cdot D'^2}$$

$$\omega_R = \frac{R \cdot D'^2}{L} \quad \omega_Z = \frac{1}{C_O \cdot R_C} \quad \omega_P = \frac{K_D}{C_O \cdot R_O} \quad \omega_L = \frac{K_m \cdot R_i}{L}$$

Current-Mode Boost – Transfer Functions in Impedance Form

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Line-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{D' \cdot \left(\frac{1}{K_m} + \frac{R_i}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Output Impedance:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Input Impedance:

$$\frac{\hat{v}_{IN}}{\hat{i}'_{IN}} = -D'^2 \cdot R \cdot \frac{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}{(G_V + K) \cdot D' - \frac{1}{K_m} \cdot \left(\frac{D'^2 \cdot R}{Z_O}\right)}$$

Control-to-Inductor Current:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{1}{R} + \frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Current Loop ($\hat{v}_C = 0$):

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Current-Mode Buck-Boost

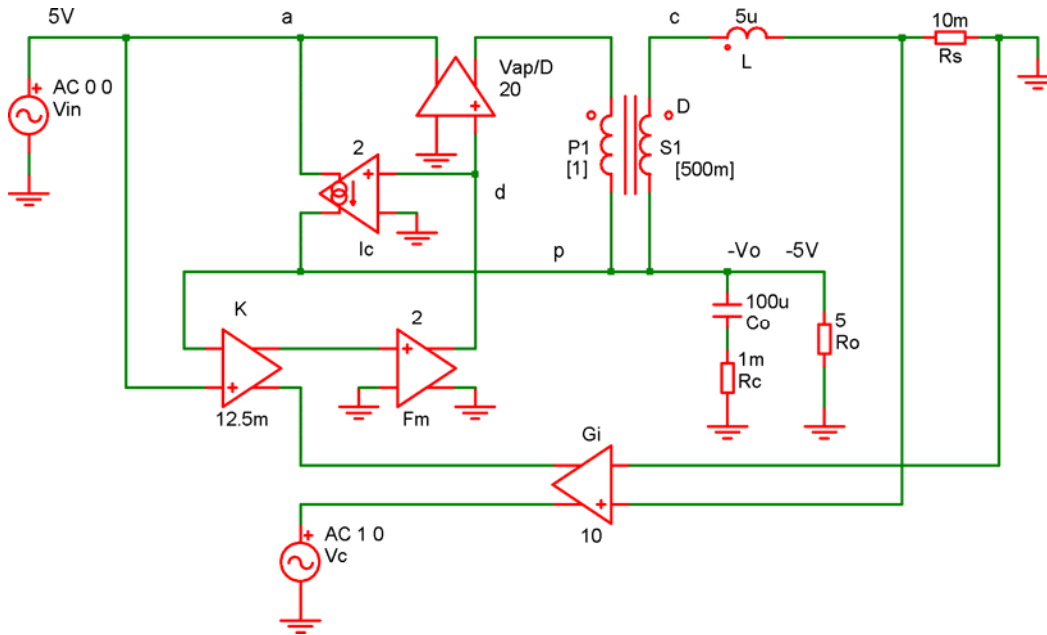


Figure 24. Buck-boost linear model

Linear Model Coefficients

$$V_{ap} = V_{IN} + V_O \quad D = \frac{V_O}{V_{IN} + V_O} \quad D' = 1 - D = \frac{V_{IN}}{V_{IN} + V_O}$$

$$M = \frac{D}{D'} \quad R = \frac{V_O}{I_O} \quad I_C = \frac{V_{ap} \cdot M}{R} \quad F_m = \frac{K_m}{V_{ap}}$$

Simplified Transfer Functions

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i \cdot K_D} \cdot \frac{\left(1 - \frac{s}{\omega_R}\right) \cdot \left(1 + \frac{s}{\omega_Z}\right)}{\left(1 + \frac{s}{\omega_P}\right) \cdot \left(1 + \frac{s}{\omega_L}\right)}$$

Line-to-Output ($\hat{v}_C = 0$):

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{R_O \cdot D \cdot D' \cdot K_n}{R_i \cdot K_D} \cdot \frac{\left(1 + \frac{s}{\omega_K}\right) \cdot \left(1 + \frac{s}{\omega_Z}\right)}{\left(1 + \frac{s}{\omega_P}\right) \cdot \left(1 + \frac{s}{\omega_L}\right)}$$

Where: $K_D = 1 + \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right)$

$K_n = \frac{1}{K_m} - \frac{K}{D} + \frac{R_i \cdot D}{R \cdot D'^2}$ $\omega_Z = \frac{1}{C_O \cdot R_C}$

$$\omega_R = \frac{R \cdot D'^2}{L \cdot D}$$

$$\omega_P = \frac{K_D}{C_O \cdot R_O}$$

$$\omega_L = \frac{K_m \cdot R_i}{L}$$

$$\omega_K = \frac{R \cdot D'^2 \cdot K_n}{L \cdot K}$$

Current-Mode Buck-Boost – Transfer Functions in Impedance Form

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Line-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{D \cdot D' \cdot \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R}\right) - K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Output Impedance:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Input Impedance:

$$\frac{\hat{v}_{IN}}{\hat{i}_{IN}} = -\frac{D'^2 \cdot R}{D^2} \cdot \frac{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\frac{K}{D} \cdot \left(D'^2 + \frac{D'^2 \cdot R + D' \cdot Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D'}{Z_O}\right) + (G_V + K) \cdot D' - \frac{1}{K_m} \cdot \left(\frac{D'^2 \cdot R}{Z_O}\right)}$$

Control-to-Inductor Current:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{D}{R} + \frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Current Loop ($\hat{v}_C = 0$):

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

General Gain Parameters

General gain parameters are listed in Table 1. These parameters are independent of topology, being written in terms of the terminal voltage V_{ap} and duty cycle D . This table has been updated to show that S_n always refers to the inductor current up-slope and S_f always refers to the inductor current down-slope. See Appendix E for a more complete unified model with revised general gain parameters for PCM2, VCM2 and EPCM3.

TABLE 1A SUMMARY OF GENERAL GAIN PARAMETERS				
		$V_{SLOPE} = S_e \cdot T$		$S_{ap} = S_n + S_f$
Mode	S_e, S_n, S_f, S_{ap}	m_c, Q	K_m, K	K_e
PCM1	$S_e = \frac{V_{SL}}{T}$ $S_n = \frac{V_{ap} \cdot D' \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_n}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D' - 0.5)}$	$K_m = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}}$ $K = 0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D'$	$K_e = 0$
PCM2	$S_e = \frac{V_{ap} \cdot D \cdot K_{SL}}{T}$ $S_n = \frac{V_{ap} \cdot D' \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_n}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D' - 0.5)}$	$K_m = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + 2 \cdot K_{SL} \cdot D}$ $K = 0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D^2$	$K_e = -K_{SL} \cdot D \cdot \frac{L}{R_i}$
VCM1	$S_e = \frac{V_{SL}}{T}$ $S_f = \frac{V_{ap} \cdot D \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_f}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D'$	$K_e = 0$
VCM2	$S_e = \frac{V_{ap} \cdot D' \cdot K_{SL}}{T}$ $S_f = \frac{V_{ap} \cdot D \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_f}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + 2 \cdot K_{SL} \cdot D'}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D'^2$	$K_e = -K_{SL} \cdot D' \cdot \frac{L}{R_i}$
EPCM1	$S_e = \frac{V_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D'$	$K_e = -D \cdot T$
EPCM2	$S_e = \frac{V_{ap} \cdot K_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D$	$K_e = -D \cdot T$

TABLE 1B SUMMARY OF GENERAL GAIN PARAMETERS		$V_{SLOPE} = S_e \cdot T$		$S_{ap} = S_n + S_f$
Mode	S_e, S_n, S_f, S_{ap}	m_c, Q	K_m, K	K_e
EVCM1	$S_e = \frac{V_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}}$ $K = 0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D'$	$K_e = -D' \cdot T$
EVCM2	$S_e = \frac{V_{ap} \cdot K_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + K_{SL}}$ $K = 0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D'$	$K_e = -D' \cdot T$
VCM3	$S_e = \frac{V_{ap} \cdot K_{SL}}{T}$ $S_f = \frac{V_{ap} \cdot D \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_f}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D'$	$K_e = 0$
EPCM3	$S_e = \frac{V_{ap} \cdot D' \cdot K_{SL} + V_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + (1 - 2 \cdot D) \cdot K_{SL} + \frac{V_{SL}}{V_{ap}}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \cdot D'$	$K_e = -D \cdot T$ $+ K_{SL} \cdot D \cdot \frac{L}{R_i}$
EPCM4	$S_e = \frac{V_{ap} \cdot K_{SL} + V_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL} + \frac{V_{SL}}{V_{ap}}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D$	$K_e = -D \cdot T$

Table Notation:

PCM – Peak Current-Mode

VCM – Valley Current-Mode

EPCM – Emulated Peak Current-Mode

EVCM – Emulated Valley Current-Mode

1 – Fixed slope compensation using V_{SL}

2 – Proportional slope compensation using K_{SL}

3, 4 – Other fixed or proportional slope compensation

Using mode 2, for $Q=0.637$ (single cycle damping), $K_{SL} = R_i \cdot \frac{T}{L}$

References

- [1] Robert Sheehan, “*Emulated Current-Mode Control for Buck Regulators Using Sample-and-Hold Technique*,” Power Electronics Technology Exhibition and Conference, PES02, October 2006.

An updated version of this paper is available from National Semiconductor Corporation which includes complete appendix material.

- [2] R.B. Ridley, “*A New, Continuous-Time Model for Current-Mode Control*,” IEEE Transactions on Power Electronics, Volume 6, Issue 2, pp. 271–280, 1991.
- [3] F.D. Tan, R.D. Middlebrook, “*A Unified Model for Current-Programmed Converters*,” IEEE Transactions on Power Electronics, Volume 10, Issue 4, pp. 397–408, 1995.

For reference [3] the following clarifications and corrections are made:

	$V_{\text{off}} = V_{\text{ap}}$	$I_{\text{on}} = I_{\text{C}}$	
Buck	$\hat{i}_1 = \hat{i}$	$L_e = L$	$E(s) = \frac{V_{\text{off}}}{D}$
Boost	$\hat{i}_1 = \hat{i}_g$	$L_e = \frac{L}{D'^2}$	$E(s) = V_{\text{off}} \cdot \left(1 - \frac{s \cdot L}{D' \cdot V_{\text{off}} / I_{\text{on}}} \right)$
Buck-Boost	$\hat{i}_1 = \hat{i} + \hat{i}_g$	$L_e = \frac{L}{D'^2}$	$E(s) = \frac{V_{\text{off}}}{D} \cdot \left(1 - \frac{s \cdot L}{D' \cdot V_{\text{off}} / I_{\text{on}}} \right)$

- [4] Robert Sheehan, “[A New Way to Model Current-Mode Control, Part 1](#),” Power Electronics Technology Magazine, May 2007.
- [5] Robert Sheehan, “[A New Way to Model Current-Mode Control, Part 2](#),” Power Electronics Technology Magazine, June 2007.

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July 31, 2007

CURRENT-MODE MODELING FOR PEAK, VALLEY AND EMULATED CONTROL METHODS

Reference Guide for Fixed-Frequency, Continuous Conduction-Mode Operation Appendix

Background

Reference [1] provides the theoretical background for this subject. The outcome of this analysis defines a single modulator gain block, which is consistent with the physical circuit. The importance of the concept of K_m as the modulator voltage gain cannot be overstated. Most linear models for current-mode control have allowed the math to define the model. In reference [1], an intuitive understanding of the modulator was used to drive the math. By algebraic manipulation, both the averaged model and continuous-time model were redefined to fit the form of the unified model. Combining the unified model gain blocks with the three-terminal PWM switch resulted in the linear models used here.

The main reference guide covers the criteria for current-mode control. From the buck regulator linear model, the gain terms are easily identified. Formal derivation of transfer functions is covered in the appendix, showing how the three-terminal switch power-stage model relates to the unified power-stage model. The unified form allows straightforward derivation for any transfer function of interest.

Transfer Functions

The low-frequency model is a second-order system, for which the equations could be written in terms of natural frequency and quality factor. Proper scaling of the current-loop gain allows the transfer functions to be shown in factored form. The inductor pole and sampling-gain term represent another second-order system, with its own natural frequency and quality factor.

In order to show a direct correlation to the linear models, a common format was chosen for the impedance form of the transfer functions. Each term is written in the most concise manner, with the gain terms from the control model appearing only once in the closed-loop form. This allows either sampling-gain term to be added at one point in either numerator or denominator.

Appendix A – Buck – Derivation of Transfer Functions

Current-Mode Buck – Power Stage

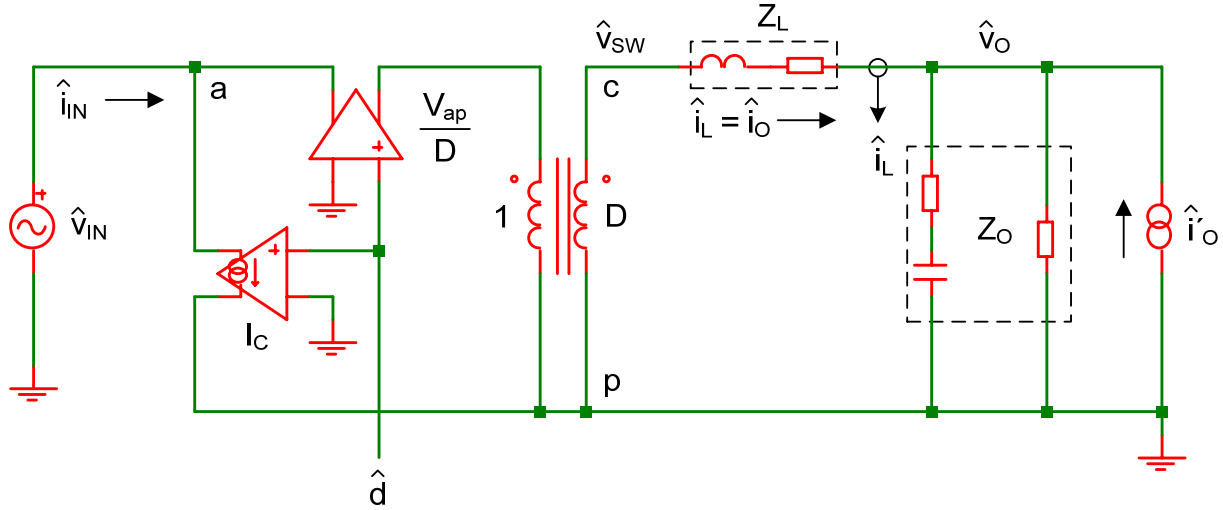


Figure A-1. Buck three-terminal switch power-stage model.

From the three-terminal switch model:

$$V_{ap} = V_{IN} \quad D = \frac{V_O}{V_{IN}} \quad D' = 1 - D = \frac{V_{IN} - V_O}{V_{IN}} \quad (\text{A.1}), (\text{A.2}), (\text{A.3})$$

$$M = D \quad R = \frac{V_O}{I_O} \quad I_C = \frac{V_{ap} \cdot M}{R} \quad (\text{A.4}), (\text{A.5}), (\text{A.6})$$

$$I_C = I_L = I_O \quad (\text{A.7})$$

Equations for Current

$$\hat{i}_C = \hat{i}_L = \hat{i}_O \quad \hat{i}_{IN} = \hat{i}_C \cdot D + I_C \cdot \hat{d} \quad (\text{A.8}), (\text{A.9})$$

By substitution, the input current is:

$$\hat{i}_{IN} = \hat{i}_L \cdot D + I_L \cdot \hat{d} \quad (\text{A.10})$$

Duty Cycle to Output Voltage

Equations for voltage in terms of current:

$$\hat{v}_{SW} = \hat{v}_O + \hat{i}_L \cdot Z_L \quad \hat{v}_O = (\hat{i}_L + \hat{i}'_O) \cdot Z_O \quad (\text{A.11}), (\text{A.12})$$

Solving A.12 for inductor current:

$$\hat{i}_L = \hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O \quad (\text{A.13})$$

From A.11 and A.13:

$$\hat{v}_{SW} = \hat{v}_O \cdot \left(1 + \frac{Z_L}{Z_O}\right) - \hat{i}'_O \cdot Z_L \quad (\text{A.14})$$

Write the equation for the switch voltage in terms of circuit voltages:

$$\hat{v}_{SW} = \left(\hat{v}_{IN} + \frac{V_{ap}}{D} \cdot \hat{d} \right) \cdot D$$
$$\hat{v}_{SW} = \hat{v}_{IN} \cdot D + \hat{d} \cdot V_{ap} \quad (\text{A.15})$$

Equate A.15 to A.14:

$$\hat{v}_{IN} \cdot D + \hat{d} \cdot V_{ap} = \hat{v}_O \cdot \left(1 + \frac{Z_L}{Z_O}\right) - \hat{i}'_O \cdot Z_L$$
$$\hat{d} \cdot V_{ap} = -\hat{v}_{IN} \cdot D + \hat{v}_O \cdot \left(1 + \frac{Z_L}{Z_O}\right) - \hat{i}'_O \cdot Z_L \quad (\text{A.16})$$

From equation A.16 with $\hat{i}'_O = 0$:

$$\hat{v}_O = \left(\hat{v}_{IN} \cdot D + \hat{d} \cdot V_{ap} \right) \cdot \left(\frac{Z_O}{Z_O + Z_L} \right) \quad (\text{A.17})$$

Equations A.10 and A.17 define the unified power-stage of reference [3]:

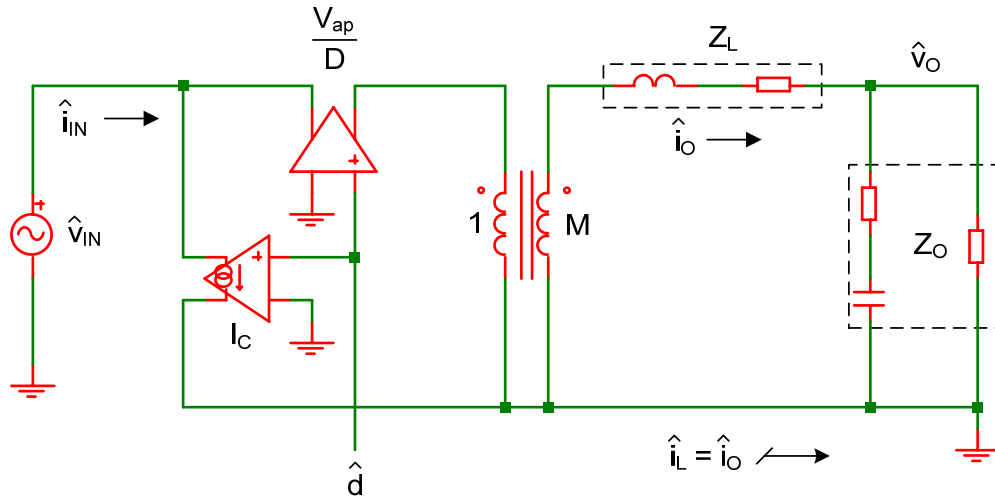


Figure A-2. Buck unified power-stage model. For the buck, this is the same as the three-terminal switch power-stage model.

Duty Cycle to Input Current and Output Voltage

From A.10:

$$I_L \cdot \hat{d} = \hat{i}_{IN} - \hat{i}_L \cdot D \quad (\text{A.18})$$

Since $\hat{i}_L = \hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O$ from A.13, and substituting for I_L :

$$\hat{d} \cdot V_{ap} \cdot \frac{D}{R} = \hat{i}_{IN} - \hat{v}_O \cdot \frac{D}{Z_O} + \hat{i}'_O \cdot D \quad (\text{A.19})$$

Duty Cycle to Inductor Current

Equate A.15 to A.11 and A.12:

$$\hat{v}_{IN} \cdot D + \hat{d} \cdot V_{ap} = (\hat{i}_L + \hat{i}'_O) \cdot Z_O + \hat{i}_L \cdot Z_L \quad (\text{A.20})$$

Solving in terms of duty cycle:

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{Z_O} = -\hat{v}_{IN} \cdot \frac{D}{Z_O} + \hat{i}_L \cdot \left(1 + \frac{Z_L}{Z_O}\right) + \hat{i}'_O \quad (\text{A.21})$$

Buck Power Stage Summary

Equations A.16, A.19 and A.21 from the power stage allow derivation of all transfer functions.

Use equation A.16 for control-to-output, line-to-output, output impedance and input impedance.
Let:

$$p_1 = V_{ap} \quad p_2 = D \quad p_3 = 1 + \frac{Z_L}{Z_O} \quad p_4 = Z_L$$

$$\hat{d} \cdot p_1 = -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4 \quad (\text{A.22})$$

Use equation A.19 for input impedance. Let $\hat{i}'_O = 0$:

$$q_1 = V_{ap} \cdot \frac{D}{R} \quad q_3 = \frac{D}{Z_O}$$

$$\hat{d} \cdot q_1 = \hat{i}_{IN} - \hat{v}_O \cdot q_3 \quad (\text{A.23})$$

Use equation A.21 for control-to-inductor current and current loop. Let $\hat{v}_{IN} = 0$ and $\hat{i}'_O = 0$:

$$u_1 = V_{ap} \cdot \frac{1}{Z_O} \quad u_5 = 1 + \frac{Z_L}{Z_O}$$

$$\hat{d} \cdot u_1 = \hat{i}_L \cdot u_5 \quad (\text{A.24})$$

Current-Mode Buck – Control Stage

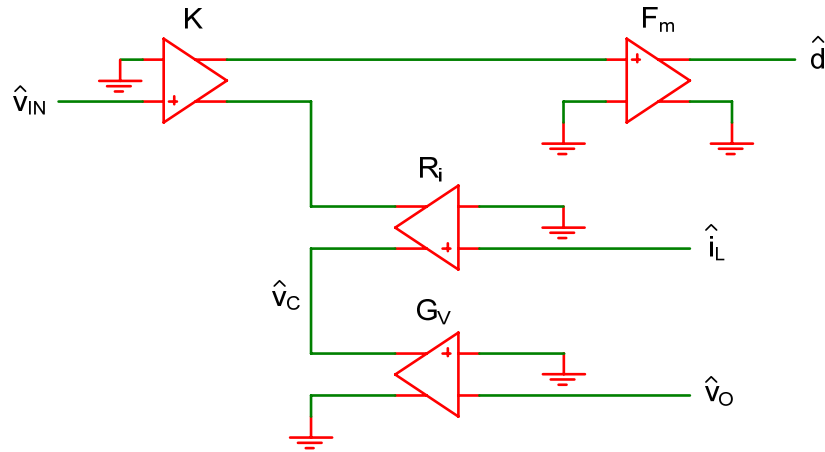


Figure A-3. Buck control-stage model.

Write the transfer function from the control-stage model:

$$\hat{d} = F_m \cdot (\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{IN} \cdot K) \quad (\text{A.25})$$

Relate inductor current to output voltage. From power-stage equation A.12:

$$\hat{i}_L = \hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O \quad (\text{A.26})$$

Substitute A.26 into A.25 to eliminate the inductor current term:

$$\hat{d} = F_m \cdot \left(\hat{v}_C - \left(\hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O \right) \cdot R_i - \hat{v}_{IN} \cdot K \right)$$

$$\hat{d} \cdot \frac{1}{F_m} = \hat{v}_C - \hat{v}_{IN} \cdot K - \hat{v}_O \cdot \frac{R_i}{Z_O} + \hat{i}'_O \cdot R_i \quad (\text{A.27})$$

Substituting for \$F_m\$:

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{K_m} = \hat{v}_C - \hat{v}_{IN} \cdot K - \hat{v}_O \cdot \frac{R_i}{Z_O} + \hat{i}'_O \cdot R_i \quad (\text{A.28})$$

For closed voltage-loop equations, let $\hat{v}_C = -G_V \cdot \hat{v}_O$. From equation A.28:

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{K_m} = -\hat{v}_{IN} \cdot K - \hat{v}_O \cdot \left(\frac{R_i}{Z_O} + G_V \right) + \hat{i}'_O \cdot R_i \quad (\text{A.29})$$

General expression from equation A.25:

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{K_m} = \hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{IN} \cdot K \quad (\text{A.30})$$

Buck Control Stage Summary

Equations A.28, A.29 and A.30 from the control stage allow derivation of all transfer functions.

Use equation A.28 for control-to-output. Let $\hat{v}_{IN} = 0$ and $\hat{i}'_O = 0$:

$$c_1 = V_{ap} \cdot \frac{1}{K_m} \quad c_3 = \frac{R_i}{Z_O}$$

$$\hat{d} \cdot c_1 = \hat{v}_C - \hat{v}_O \cdot c_3 \quad (\text{A.31})$$

Use equation A.29 for line-to-output, output impedance and input impedance. Let:

$$c_1 = V_{ap} \cdot \frac{1}{K_m} \quad c_2 = K \quad c_{3A} = \frac{R_i}{Z_O} + G_V \quad c_4 = R_i$$

$$\hat{d} \cdot c_1 = -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} + \hat{i}'_O \cdot c_4 \quad (\text{A.32})$$

Use equation A.30 for control-to-inductor current with $\hat{v}_{IN} = 0$. Let:

$$b_1 = V_{ap} \cdot \frac{1}{K_m} \quad b_5 = R_i$$

$$\hat{d} \cdot b_1 = \hat{v}_C - \hat{i}_L \cdot b_5 \quad (\text{A.33})$$

Use equation A.30 for the current loop with $\hat{v}_C = 0$ and $\hat{v}_{IN} = 0$. Designate \hat{i}'_L as the control input:

$$b_1 = V_{ap} \cdot \frac{1}{K_m} \quad b_5 = R_i$$

$$\hat{d} \cdot b_1 = -\hat{i}'_L \cdot b_5 \quad (\text{A.34})$$

Buck – Control-to-Output

From equation A.22 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation A.31:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= \hat{v}_C - \hat{v}_O \cdot c_3 \\ p_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) &= c_1 \cdot (\hat{v}_O \cdot p_3) \\ \frac{\hat{v}_O}{\hat{v}_C} &= \frac{p_1}{c_1 \cdot p_3 + c_3 \cdot p_1}\end{aligned}\tag{A.35}$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{1}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + \frac{R_i}{Z_O}}$$

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{1}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right)}\tag{A.36}$$

Buck – Line-to-Output

From equations A.22 and A.32 with $\hat{i}'_O = 0$:

$$\begin{aligned} \hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} \\ p_1 \cdot (-\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A}) &= c_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \frac{\hat{v}_O}{\hat{v}_{IN}} &= \frac{c_1 \cdot p_2 - c_2 \cdot p_1}{c_1 \cdot p_3 + c_{3A} \cdot p_1} \end{aligned} \quad (\text{A.37})$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{\frac{1}{K_m} \cdot D - K}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + \frac{R_i}{Z_O} + G_V}$$

Line-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{D \cdot \left(\frac{1}{K_m} - \frac{K}{D}\right)}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right) + G_V} \quad (\text{A.38})$$

Buck – Output Impedance

From equations A.22 and A.32 with $\hat{v}_{IN} = 0$:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4 & \hat{d} \cdot c_1 &= -\hat{v}_O \cdot c_{3A} + \hat{i}'_O \cdot c_4 \\ p_1 \cdot (-\hat{v}_O \cdot c_{3A} + \hat{i}'_O \cdot c_4) &= c_1 \cdot (\hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4) \\ \frac{\hat{v}_O}{\hat{i}'_O} &= \frac{c_1 \cdot p_4 + c_4 \cdot p_1}{c_1 \cdot p_3 + c_{3A} \cdot p_1}\end{aligned}\tag{A.39}$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + \frac{R_i}{Z_O} + G_V}$$

Output Impedance:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right) + G_V}\tag{A.40}$$

Buck – Input Impedance

From equations A.22 and A.32 with $\hat{i}'_O = 0$:

$$\begin{aligned}\hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} \\ p_1 \cdot (-\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A}) &= c_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (c_1 \cdot p_3 + c_{3A} \cdot p_1) &= \hat{v}_{IN} \cdot (c_1 \cdot p_2 - c_2 \cdot p_1)\end{aligned}\tag{A.41}$$

From equation A.22 with $\hat{i}'_O = 0$ and equation A.23:

$$\begin{aligned}\hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot q_1 &= \hat{i}_{IN} - \hat{v}_O \cdot q_3 \\ p_1 \cdot (\hat{i}_{IN} - \hat{v}_O \cdot q_3) &= q_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (p_3 \cdot q_1 + p_1 \cdot q_3) &= \hat{v}_{IN} \cdot p_2 \cdot q_1 + \hat{i}_{IN} \cdot p_1\end{aligned}\tag{A.42}$$

Equate A.41 to A.42:

$$\begin{aligned}(\hat{v}_{IN} \cdot p_2 \cdot q_1 + \hat{i}_{IN} \cdot p_1) \cdot (c_1 \cdot p_3 + c_{3A} \cdot p_1) &= \hat{v}_{IN} \cdot (c_1 \cdot p_2 - c_2 \cdot p_1) \cdot (p_3 \cdot q_1 + p_1 \cdot q_3) \\ \frac{\hat{v}_{IN}}{\hat{i}_{IN}} &= -\frac{c_1 \cdot p_3 + c_{3A} \cdot p_1}{c_2 \cdot p_3 \cdot q_1 + c_{3A} \cdot p_2 \cdot q_1 + c_2 \cdot p_1 \cdot q_3 - c_1 \cdot p_2 \cdot q_3} \\ \frac{\hat{v}_{IN}}{\hat{i}_{IN}} &= -\frac{c_1 \cdot p_3 + c_{3A} \cdot p_1}{c_2 \cdot (p_3 \cdot q_1 + p_1 \cdot q_3) + p_2 \cdot (c_{3A} \cdot q_1 - c_1 \cdot q_3)}\end{aligned}\tag{A.43}$$

Substitute for variables:

Numerator:

$$\frac{c_1 \cdot p_3 + c_{3A} \cdot p_1}{V_{ap}} = \frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O} \right) + \frac{R_i}{Z_O} + G_v$$

Buck – Input Impedance (continued)

Denominator:

$$\frac{c_2 \cdot (p_3 \cdot q_1 + p_1 \cdot q_3)}{V_{ap}} = K \cdot \left(\left(1 + \frac{Z_L}{Z_O} \right) \cdot \frac{D}{R} + \frac{D}{Z_O} \right)$$

$$\frac{p_2 \cdot (c_{3A} \cdot q_1 - c_1 \cdot q_3)}{V_{ap}} = D \cdot \left(\left(\frac{R_i}{Z_O} + G_V \right) \cdot \frac{D}{R} - \frac{1}{K_m} \cdot \frac{D}{Z_O} \right)$$

$$\frac{c_2 \cdot (p_3 \cdot q_1 + p_1 \cdot q_3)}{V_{ap}} = \frac{K \cdot D}{R} \cdot \left(1 + \frac{Z_L}{Z_O} + \frac{R}{Z_O} \right)$$

$$\frac{p_2 \cdot (c_{3A} \cdot q_1 - c_1 \cdot q_3)}{V_{ap}} = \frac{D^2}{R} \cdot \left(\frac{R_i}{Z_O} + G_V - \frac{1}{K_m} \cdot \frac{R}{Z_O} \right)$$

Input Impedance:

$$\frac{\hat{v}_{IN}}{\hat{i}_{IN}} = -\frac{R}{D^2} \cdot \frac{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O} \right) + R_i \cdot \left(\frac{1}{Z_O} \right) + G_V}{\frac{K}{D} \cdot \left(1 + \frac{R + Z_L}{Z_O} \right) + R_i \cdot \left(\frac{1}{Z_O} \right) + G_V - \frac{1}{K_m} \cdot \left(\frac{R}{Z_O} \right)} \quad (\text{A.44})$$

Buck – Control-to-Inductor Current

From equations A.24 and A.33:

$$\begin{aligned}\hat{d} \cdot u_1 &= \hat{i}_L \cdot u_5 & \hat{d} \cdot b_1 &= \hat{v}_C - \hat{i}_L \cdot b_5 \\ u_1 \cdot (\hat{v}_C - \hat{i}_L \cdot b_5) &= b_1 \cdot (\hat{i}_L \cdot u_5) \\ \frac{\hat{i}_L}{\hat{v}_C} &= \frac{u_1}{b_1 \cdot u_5 + b_5 \cdot u_1}\end{aligned}\tag{A.45}$$

Substitute for variables:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \frac{1}{Z_O}}$$

Control-to-Inductor Current:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{Z_O}\right)}\tag{A.46}$$

Buck – Current Loop

From equations A.24 and A.34:

$$\begin{aligned}\hat{d} \cdot u_1 &= \hat{i}_L \cdot u_5 & \hat{d} \cdot b_1 &= -\hat{i}'_L \cdot b_5 \\ u_1 \cdot (-\hat{i}'_L \cdot b_5) &= b_1 \cdot (\hat{i}_L \cdot u_5) \\ \frac{\hat{i}_L}{\hat{i}'_L} &= \frac{-b_5 \cdot u_1}{u_5 \cdot b_1}\end{aligned}\tag{A.47}$$

Substitute for variables:

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \frac{1}{Z_O}}{\left(1 + \frac{Z_L}{Z_O}\right) \cdot \frac{1}{K_m}}$$

Current Loop ($\hat{v}_C = 0$):

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\frac{1}{Z_O}\right)}{\frac{1}{K_m} \cdot \left(1 + \frac{Z_L}{Z_O}\right)}\tag{A.48}$$

Buck – Derivation of Factored Form

Starting with the control-to-output gain of equation A.36, regroup and factor the dc terms:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O}{R_i} \cdot \frac{\frac{Z_O}{R_O}}{1 + \frac{Z_O + Z_L}{K_m \cdot R_i}}$$

Expand Z_O and Z_L :

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O}{R_i} \cdot \frac{1 + s \cdot C_O \cdot R_C}{1 + s \cdot C_O \cdot (R_O + R_C) + \frac{R_O \cdot (1 + s \cdot C_O \cdot R_C) + (s \cdot L + R_L + R_S) \cdot (1 + s \cdot C_O \cdot (R_O + R_C))}{K_m \cdot R_i}}$$

Expand the denominator and group like terms:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O}{R_i} \cdot \frac{1 + s \cdot C_O \cdot R_C}{K_D + s \cdot C_O \cdot K_C + \left(\frac{s \cdot L}{K_m \cdot R_i} \right) \cdot (1 + s \cdot C_O \cdot (R_O + R_C))} \quad (\text{A.49})$$

Where:

$$K_D = 1 + \frac{R_O + R_L + R_S}{K_m \cdot R_i} \quad K_C = (R_O + R_C) \cdot \left(1 + \frac{R_L + R_S + \frac{R_O \cdot R_C}{R_O + R_C}}{K_m \cdot R_i} \right) \quad (\text{A.50})$$

By factoring this becomes:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O}{R_i \cdot K_D} \cdot \frac{1 + s \cdot C_O \cdot R_C}{\left[1 + s \cdot C_O \cdot \frac{K_C}{K_D} \right] \cdot \left[1 + \left(\frac{s \cdot L}{K_m \cdot R_i} \right) \cdot \left(\frac{1 + s \cdot C_O \cdot (R_O + R_C)}{K_D + s \cdot C_O \cdot K_C} \right) \right]} \quad (\text{A.51})$$

$$\text{To simplify: } K_D \approx 1 + \frac{R_O}{K_m \cdot R_i} \quad K_C \approx R_O \quad \frac{1 + s \cdot C_O \cdot (R_O + R_C)}{K_D + s \cdot C_O \cdot K_C} \approx 1 \quad (\text{A.52})$$

The simplified expression is:

$$\frac{\hat{v}_O}{\hat{v}_C} \approx \frac{R_O}{R_i \cdot K_D} \cdot \frac{1 + s \cdot C_O \cdot R_C}{\left(1 + s \cdot \frac{C_O \cdot R_O}{K_D} \right) \cdot \left(1 + s \cdot \frac{L}{K_m \cdot R_i} \right)} \quad (\text{A.53})$$

Buck – Linear Equation Results

Results from the linear equations were compared to the linear model transfer functions, showing excellent agreement. No sampling-gain term was used for the comparison. For the output impedance and input impedance, $G_V = 3$ represents the typical mid-band error amplifier gain.

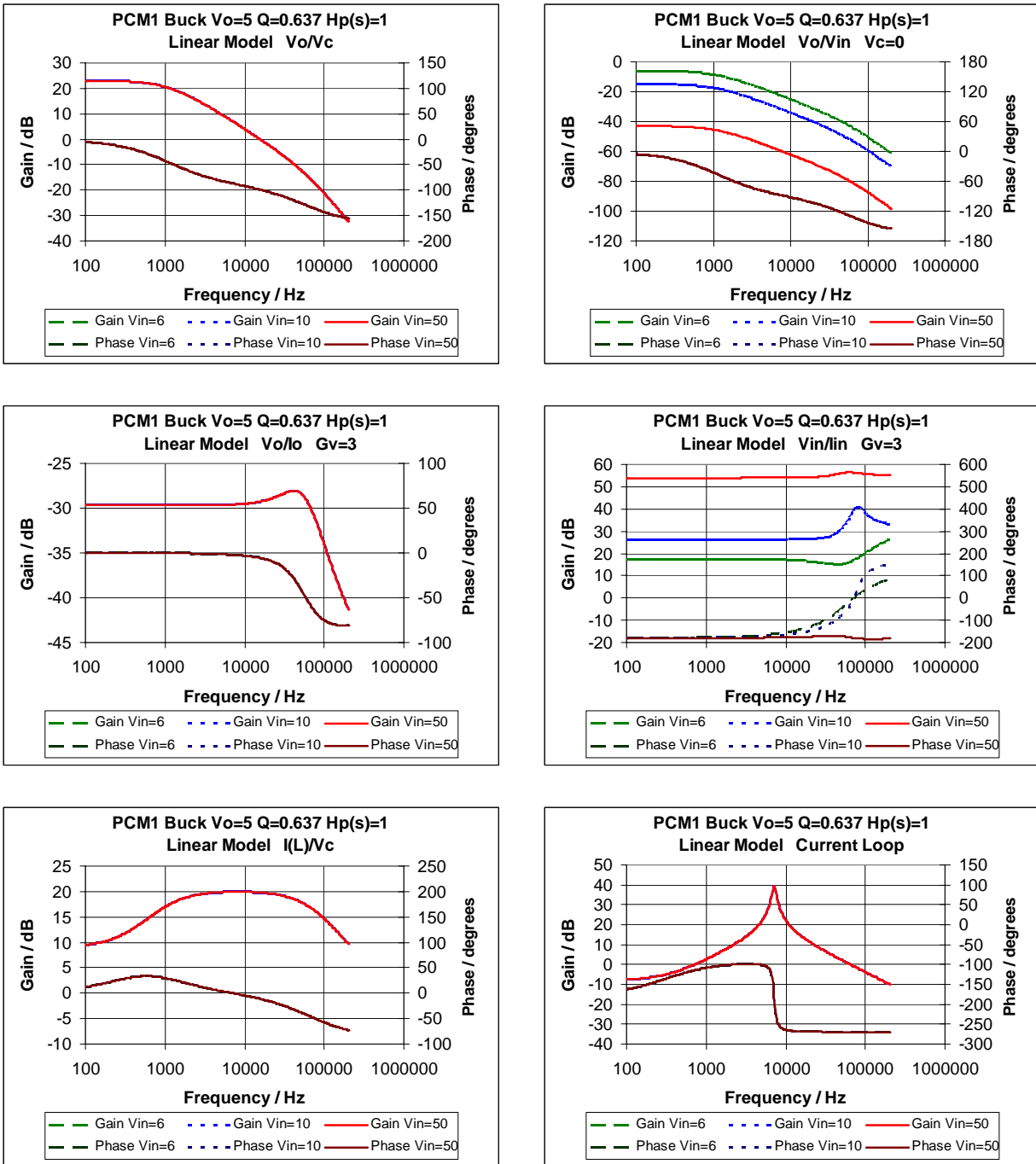


Figure A-4. Buck linear equation results, without sampling-gain term.

Appendix B – Boost – Derivation of Transfer Functions

Current-Mode Boost – Power Stage

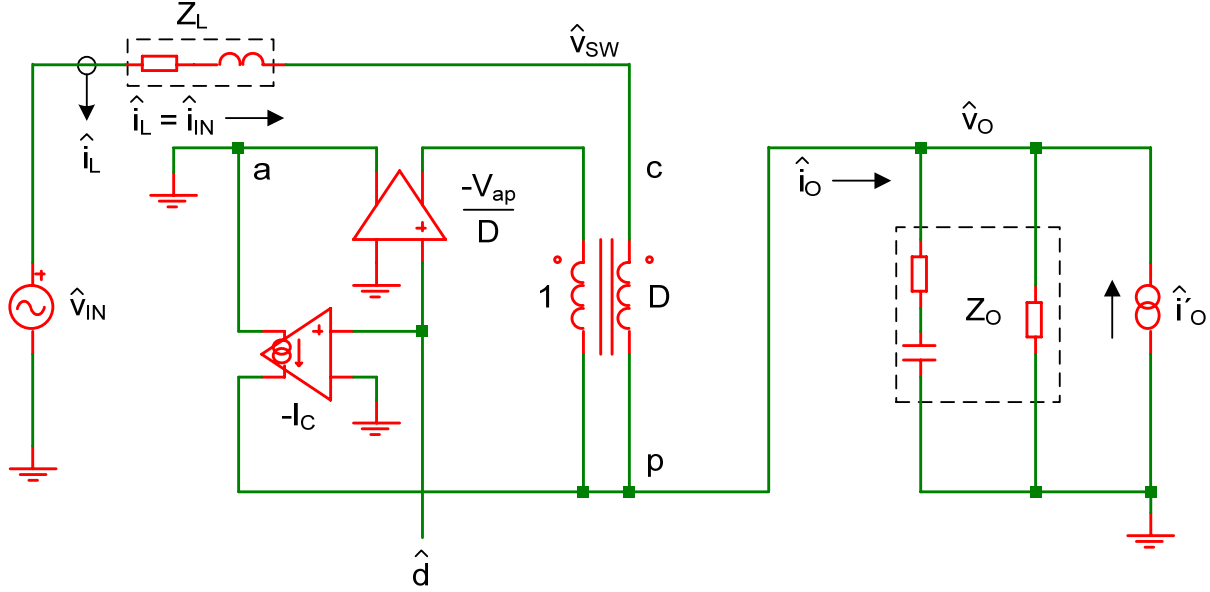


Figure B-1. Boost three-terminal switch power-stage model.

From the three-terminal switch model:

$$V_{ap} = V_O \quad D = \frac{V_O - V_{IN}}{V_O} \quad D' = 1 - D = \frac{V_{IN}}{V_O} \quad (\text{B.1}), (\text{B.2}), (\text{B.3})$$

$$M = \frac{1}{D'} \quad R = \frac{V_O}{I_O} \quad I_C = \frac{V_{ap} \cdot M}{R} \quad -I_C = I_L = I_{IN} \quad (\text{B.4}), (\text{B.5}), (\text{B.6}), (\text{B.7})$$

Equations for Current

$$-\hat{i}_C = \hat{i}_L = \hat{i}_{IN} \quad \hat{i}_O + \hat{i}_C = \hat{i}_C \cdot D + I_C \cdot \hat{d} \quad (\text{B.8}), (\text{B.9})$$

By substitution:

$$\hat{i}_O - \hat{i}_L = -\hat{i}_L \cdot D - I_L \cdot \hat{d} \quad (\text{B.10})$$

This allows the inductor current to be expressed as:

$$\hat{i}_L = \hat{d} \cdot \frac{I_L}{D'} + \hat{i}_O \cdot \frac{1}{D'} \quad (\text{B.11})$$

Duty Cycle to Output Voltage

Equations for voltage in terms of current:

$$\hat{v}_{IN} - \hat{v}_{SW} = \hat{i}_L \cdot Z_L \quad \hat{v}_O = (\hat{i}_O + \hat{i}'_O) \cdot Z_O \quad (\text{B.12}), (\text{B.13})$$

Solving B.13 for output current:

$$\hat{i}_O = \hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O \quad (\text{B.14})$$

From equations B.11 and B.12:

$$\hat{v}_{IN} - \hat{v}_{SW} = \left(\hat{d} \cdot \frac{I_L}{D'} + \hat{i}_O \cdot \frac{1}{D'} \right) \cdot Z_L \quad (\text{B.15})$$

Substitute B.13 into B.14:

$$\hat{v}_{IN} - \hat{v}_{SW} = \left(\hat{d} \cdot \frac{I_L}{D'} + \left(\hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O \right) \cdot \frac{1}{D'} \right) \cdot Z_L \quad (\text{B.16})$$

Solving B.16 for switch voltage:

$$\hat{v}_{SW} = \hat{v}_{IN} - \hat{d} \cdot \frac{I_L \cdot Z_L}{D'} - \hat{v}_O \cdot \frac{Z_L}{D' \cdot Z_O} + \hat{i}'_O \cdot \frac{Z_L}{D'} \quad (\text{B.17})$$

Write the equation for the switch voltage in terms of circuit voltages:

$$\hat{v}_{SW} = \hat{v}_O + \left(-\frac{V_{ap}}{D} \cdot \hat{d} - \hat{v}_O \right) \cdot D$$

$$\hat{v}_{SW} = -\hat{d} \cdot V_{ap} + \hat{v}_O \cdot D' \quad (\text{B.18})$$

Equate B.17 to B.18:

$$\hat{v}_{IN} - \hat{d} \cdot \frac{I_L \cdot Z_L}{D'} - \hat{v}_O \cdot \frac{Z_L}{D' \cdot Z_O} + \hat{i}'_O \cdot \frac{Z_L}{D'} = -\hat{d} \cdot V_{ap} + \hat{v}_O \cdot D'$$

$$\hat{d} \cdot D' \cdot \left(V_{ap} - \frac{I_L \cdot Z_L}{D'} \right) = -\hat{v}_{IN} \cdot D' + \hat{v}_O \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) - \hat{i}'_O \cdot Z_L \quad (\text{B.19})$$

Substituting for I_L :

$$\hat{d} \cdot V_{ap} \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right) = -\hat{v}_{IN} \cdot D' + \hat{v}_O \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) - \hat{i}'_O \cdot Z_L \quad (\text{B.20})$$

From equation B.20 with $\hat{i}'_O = 0$:

$$\hat{v}_O = \left(\hat{v}_{IN} \cdot \frac{1}{D'} + \hat{d} \cdot \frac{V_{ap}}{D'} \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right) \right) \cdot \left(\frac{Z_O}{Z_O + \frac{Z_L}{D'^2}} \right) \quad (\text{B.21})$$

Equations B.11 and B.21 define the unified power stage of reference [3]:

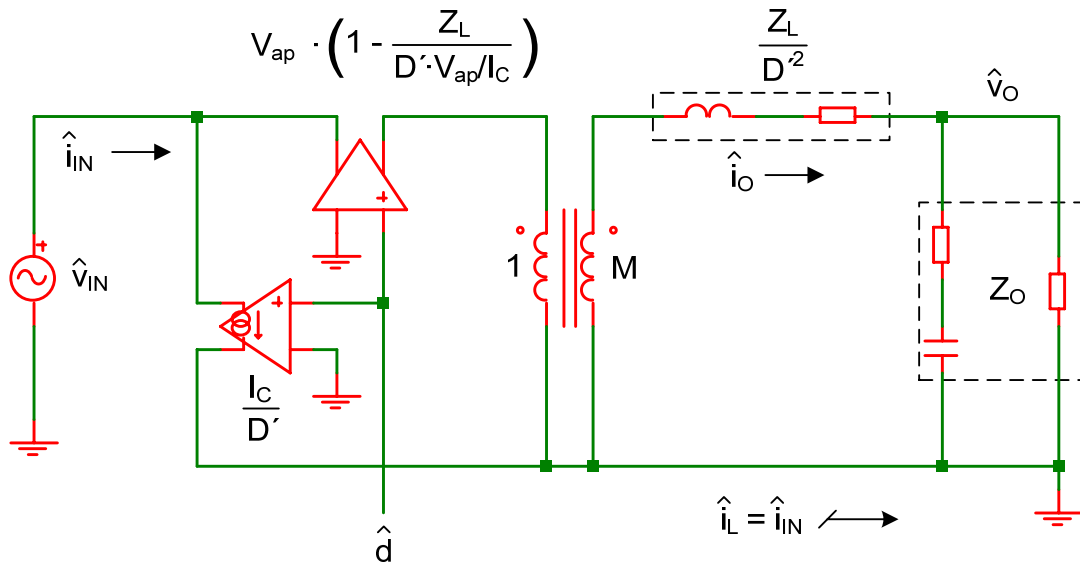


Figure B-2. Boost unified power-stage model, where $|I_c| = |I_L|$.

Duty Cycle to Inductor Current and Output Voltage

From B.12:

$$\hat{v}_{SW} = \hat{v}_{IN} - \hat{i}_L \cdot Z_L \quad (\text{B.22})$$

From B.22 and B.18:

$$\hat{v}_{IN} - \hat{i}_L \cdot Z_L = -\hat{d} \cdot V_{ap} + \hat{v}_O \cdot D' \quad (\text{B.23})$$

From B.23:

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{Z_L} = -\hat{v}_{IN} \cdot \frac{1}{Z_L} + \hat{i}_L + \hat{v}_O \cdot \frac{D'}{Z_L} \quad (\text{B.24})$$

Boost Power Stage Summary

Equations B.20 and B.24 from the power stage allow derivation of all transfer functions.

Use equation B.20 for all transfer functions. Let:

$$p_1 = V_{ap} \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right) \quad p_2 = D' \quad p_3 = D'^2 + \frac{Z_L}{Z_O} \quad p_4 = Z_L$$

$$\hat{d} \cdot p_1 = -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4 \quad (\text{B.25})$$

Use equation B.24 for input impedance, control-to-inductor current and current loop. Let:

$$u_1 = V_{ap} \cdot \frac{1}{Z_L} \quad u_2 = \frac{1}{Z_L} \quad u_3 = \frac{D'}{Z_L}$$

$$\hat{d} \cdot u_1 = -\hat{v}_{IN} \cdot u_2 + \hat{i}_L + \hat{v}_O \cdot u_3 \quad (\text{B.26})$$

Current-Mode Boost – Control Stage

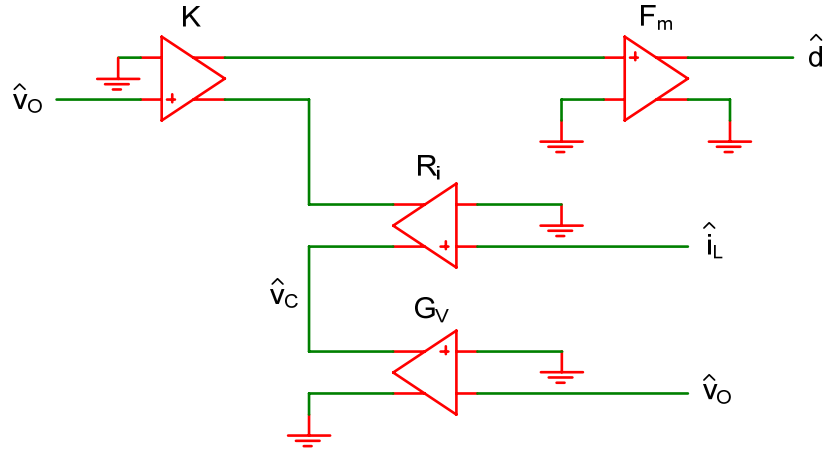


Figure B-3. Boost control-stage model.

Write the transfer function from the control-stage model:

$$\hat{d} = F_m \cdot (\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_O \cdot K) \quad (\text{B.27})$$

Relate inductor current to output voltage. From power-stage equation B.23:

$$\hat{i}_L = \hat{v}_{IN} \cdot \frac{1}{Z_L} + \hat{d} \cdot \frac{V_{ap}}{Z_L} - \hat{v}_O \cdot \frac{D'}{Z_L} \quad (\text{B.28})$$

Substitute B.28 into B.27 to eliminate the inductor current term:

$$\begin{aligned} \hat{d} &= F_m \cdot \left(\hat{v}_C - \left(\hat{v}_{IN} \cdot \frac{1}{Z_L} + \hat{d} \cdot \frac{V_{ap}}{Z_L} - \hat{v}_O \cdot \frac{D'}{Z_L} \right) \cdot R_i - \hat{v}_O \cdot K \right) \\ \hat{d} \cdot \left(\frac{1}{F_m} + \frac{V_{ap} \cdot R_i}{Z_L} \right) &= \hat{v}_C - \hat{v}_{IN} \cdot \frac{R_i}{Z_L} - \hat{v}_O \cdot \left(K - \frac{D' \cdot R_i}{Z_L} \right) \end{aligned} \quad (\text{B.29})$$

Factoring V_{ap} :

$$\hat{d} \cdot V_{ap} \cdot \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) = \hat{v}_C - \hat{v}_{IN} \cdot \frac{R_i}{Z_L} - \hat{v}_O \cdot \left(K - \frac{D' \cdot R_i}{Z_L} \right) \quad (\text{B.30})$$

For closed voltage-loop equations, let $\hat{v}_C = -G_V \cdot \hat{v}_O$. From equation B.30:

$$\hat{d} \cdot V_{ap} \cdot \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) = -\hat{v}_{IN} \cdot \frac{R_i}{Z_L} - \hat{v}_O \cdot \left(G_V + K - \frac{D' \cdot R_i}{Z_L} \right) \quad (\text{B.31})$$

General expression from equation B.27

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{K_m} = \hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_O \cdot K \quad (\text{B.32})$$

Boost Control Stage Summary

Equations B.30, B.31 and B.32 from the control stage allow derivation of all transfer functions.

Use equation B.30 for control-to-output and control-to-inductor current with $\hat{v}_{IN} = 0$. Let:

$$c_1 = V_{ap} \cdot \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \quad c_3 = K - \frac{D' \cdot R_i}{Z_L}$$

$$\hat{d} \cdot c_1 = \hat{v}_C - \hat{v}_O \cdot c_3 \quad (\text{B.33})$$

Use equation B.31 for line-to-output, output impedance and input impedance. Let:

$$c_1 = V_{ap} \cdot \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \quad c_2 = \frac{R_i}{Z_L} \quad c_{3A} = G_V + K - \frac{D' \cdot R_i}{Z_L}$$

$$\hat{d} \cdot c_1 = -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} \quad (\text{B.34})$$

Use equation B.32 for the current loop with $\hat{v}_C = 0$. Designate \hat{i}'_L as the control input:

$$b_1 = V_{ap} \cdot \frac{1}{K_m} \quad b_5 = R_i \quad b_3 = K$$

$$\hat{d} \cdot b_1 = -\hat{i}'_L \cdot b_5 - \hat{v}_O \cdot b_3 \quad (\text{B.35})$$

Boost – Control-to-Output

From equation B.25 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation B.33:

$$\hat{d} \cdot p_1 = \hat{v}_O \cdot p_3 \quad \hat{d} \cdot c_1 = \hat{v}_C - \hat{v}_O \cdot c_3$$

$$p_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) = c_1 \cdot (\hat{v}_O \cdot p_3)$$

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{p_1}{c_1 \cdot p_3 + c_3 \cdot p_1} \quad (\text{B.36})$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}{\left(\frac{1}{K_m} + \frac{R_i}{Z_L}\right) \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + \left(K - \frac{D' \cdot R_i}{Z_L}\right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)} \quad (\text{B.37})$$

Boost – Line-to-Output

From equation B.25 with $\hat{i}'_O = 0$ and equation B.34:

$$\begin{aligned} \hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} \\ p_1 \cdot (-\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A}) &= c_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \frac{\hat{v}_O}{\hat{v}_{IN}} &= \frac{c_1 \cdot p_2 - c_2 \cdot p_1}{c_1 \cdot p_3 + c_{3A} \cdot p_1} \end{aligned} \quad (\text{B.38})$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{\left(\frac{1}{K_m} + \frac{R_i}{Z_L}\right) \cdot D' - \left(\frac{R_i}{Z_L}\right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}{\left(\frac{1}{K_m} + \frac{R_i}{Z_L}\right) \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + \left(G_V + K - \frac{D' \cdot R_i}{Z_L}\right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Line-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{D' \cdot \left(\frac{1}{K_m} + \frac{R_i}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)} \quad (\text{B.39})$$

Boost – Output Impedance

From equations B.25 and B.34 with $\hat{v}_{IN} = 0$:

$$\hat{d} \cdot p_1 = \hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4 \quad \hat{d} \cdot c_1 = -\hat{v}_O \cdot c_{3A}$$

$$p_1 \cdot (-\hat{v}_O \cdot c_{3A}) = c_1 \cdot (\hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4)$$

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{c_1 \cdot p_4}{c_1 \cdot p_3 + c_{3A} \cdot p_1} \quad (\text{B.40})$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \cdot Z_L}{\left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + \left(G_V + K - \frac{D' \cdot R_i}{Z_L} \right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)}$$

Output Impedance:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O} \right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)} \quad (\text{B.41})$$

Boost – Input Impedance

From equation B.25 with $\hat{i}'_O = 0$ and equation B.34:

$$\begin{aligned}\hat{\mathbf{d}} \cdot \mathbf{p}_1 &= -\hat{\mathbf{v}}_{IN} \cdot \mathbf{p}_2 + \hat{\mathbf{v}}_O \cdot \mathbf{p}_3 & \hat{\mathbf{d}} \cdot \mathbf{c}_1 &= -\hat{\mathbf{v}}_{IN} \cdot \mathbf{c}_2 - \hat{\mathbf{v}}_O \cdot \mathbf{c}_{3A} \\ \mathbf{p}_1 \cdot (-\hat{\mathbf{v}}_{IN} \cdot \mathbf{c}_2 - \hat{\mathbf{v}}_O \cdot \mathbf{c}_{3A}) &= \mathbf{c}_1 \cdot (-\hat{\mathbf{v}}_{IN} \cdot \mathbf{p}_2 + \hat{\mathbf{v}}_O \cdot \mathbf{p}_3) \\ \hat{\mathbf{v}}_O \cdot (\mathbf{c}_1 \cdot \mathbf{p}_3 + \mathbf{c}_{3A} \cdot \mathbf{p}_1) &= \hat{\mathbf{v}}_{IN} \cdot (\mathbf{c}_1 \cdot \mathbf{p}_2 - \mathbf{c}_2 \cdot \mathbf{p}_1)\end{aligned}\tag{B.42}$$

From equation B.25 with $\hat{i}'_O = 0$ and equation B.26:

$$\begin{aligned}\hat{\mathbf{d}} \cdot \mathbf{p}_1 &= -\hat{\mathbf{v}}_{IN} \cdot \mathbf{p}_2 + \hat{\mathbf{v}}_O \cdot \mathbf{p}_3 & \hat{\mathbf{d}} \cdot \mathbf{u}_1 &= -\hat{\mathbf{v}}_{IN} \cdot \mathbf{u}_2 + \hat{\mathbf{i}}_L + \hat{\mathbf{v}}_O \cdot \mathbf{u}_3 \\ \mathbf{p}_1 \cdot (-\hat{\mathbf{v}}_{IN} \cdot \mathbf{u}_2 + \hat{\mathbf{i}}_L + \hat{\mathbf{v}}_O \cdot \mathbf{u}_3) &= \mathbf{u}_1 \cdot (-\hat{\mathbf{v}}_{IN} \cdot \mathbf{p}_2 + \hat{\mathbf{v}}_O \cdot \mathbf{p}_3) \\ \hat{\mathbf{v}}_O \cdot (\mathbf{p}_3 \cdot \mathbf{u}_1 - \mathbf{p}_1 \cdot \mathbf{u}_3) &= \hat{\mathbf{v}}_{IN} \cdot (\mathbf{p}_2 \cdot \mathbf{u}_1 - \mathbf{p}_1 \cdot \mathbf{u}_2) + \hat{\mathbf{i}}_L \cdot \mathbf{p}_1\end{aligned}\tag{B.43}$$

Equate B.42 to B.43:

$$\begin{aligned}(\hat{\mathbf{v}}_{IN} \cdot (\mathbf{p}_2 \cdot \mathbf{u}_1 - \mathbf{p}_1 \cdot \mathbf{u}_2) + \hat{\mathbf{i}}_L \cdot \mathbf{p}_1) \cdot (\mathbf{c}_1 \cdot \mathbf{p}_3 + \mathbf{c}_{3A} \cdot \mathbf{p}_1) &= \hat{\mathbf{v}}_{IN} \cdot (\mathbf{c}_1 \cdot \mathbf{p}_2 - \mathbf{c}_2 \cdot \mathbf{p}_1) \cdot (\mathbf{p}_3 \cdot \mathbf{u}_1 - \mathbf{p}_1 \cdot \mathbf{u}_3) \\ \frac{\hat{\mathbf{v}}_{IN}}{\hat{\mathbf{i}}_L} &= -\frac{\mathbf{c}_1 \cdot \mathbf{p}_3 + \mathbf{c}_{3A} \cdot \mathbf{p}_1}{\mathbf{c}_1 \cdot \mathbf{p}_2 \cdot \mathbf{u}_3 + \mathbf{c}_{3A} \cdot \mathbf{p}_2 \cdot \mathbf{u}_1 + \mathbf{c}_2 \cdot \mathbf{p}_3 \cdot \mathbf{u}_1 - \mathbf{c}_2 \cdot \mathbf{p}_1 \cdot \mathbf{u}_3 - \mathbf{c}_1 \cdot \mathbf{p}_3 \cdot \mathbf{u}_2 - \mathbf{c}_{3A} \cdot \mathbf{p}_1 \cdot \mathbf{u}_2} \\ \frac{\hat{\mathbf{v}}_{IN}}{\hat{\mathbf{i}}_L} &= -\frac{\mathbf{c}_1 \cdot \mathbf{p}_3 + \mathbf{c}_{3A} \cdot \mathbf{p}_1}{\mathbf{c}_1 \cdot (\mathbf{p}_2 \cdot \mathbf{u}_3 - \mathbf{p}_3 \cdot \mathbf{u}_2) + \mathbf{c}_2 \cdot (\mathbf{p}_3 \cdot \mathbf{u}_1 - \mathbf{p}_1 \cdot \mathbf{u}_3) + \mathbf{c}_{3A} \cdot (\mathbf{p}_2 \cdot \mathbf{u}_1 - \mathbf{p}_1 \cdot \mathbf{u}_2)}\end{aligned}\tag{B.44}$$

Substitute for variables with $\hat{\mathbf{i}}_{IN} = \hat{\mathbf{i}}_L$:

Numerator:

$$\frac{\mathbf{c}_1 \cdot \mathbf{p}_3 + \mathbf{c}_{3A} \cdot \mathbf{p}_1}{V_{ap}} = \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + \left(G_v + K - \frac{D' \cdot R_i}{Z_L} \right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)$$

Boost – Input Impedance (continued)

Denominator:

$$\frac{c_1 \cdot (p_2 \cdot u_3 - p_3 \cdot u_2)}{V_{ap}} = \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \cdot \left(D' \cdot \frac{D'}{Z_L} - \left(D'^2 + \frac{Z_L}{Z_O} \right) \cdot \frac{1}{Z_L} \right)$$

$$\frac{c_2 \cdot (p_3 \cdot u_1 - p_1 \cdot u_3)}{V_{ap}} = \left(\frac{R_i}{Z_L} \right) \cdot \left(\left(D'^2 + \frac{Z_L}{Z_O} \right) \cdot \frac{1}{Z_L} - D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right) \cdot \frac{D'}{Z_L} \right)$$

$$\frac{c_{3A} \cdot (p_2 \cdot u_1 - p_1 \cdot u_2)}{V_{ap}} = \left(G_v + K - \frac{D' \cdot R_i}{Z_L} \right) \cdot \left(D' \cdot \frac{1}{Z_L} - D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right) \cdot \frac{1}{Z_L} \right)$$

$$\frac{c_1 \cdot (p_2 \cdot u_3 - p_3 \cdot u_2)}{V_{ap}} = \left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \cdot \left(-\frac{1}{Z_O} \right)$$

$$\frac{c_2 \cdot (p_3 \cdot u_1 - p_1 \cdot u_3)}{V_{ap}} = \left(\frac{R_i}{Z_L} \right) \cdot \left(\frac{1}{Z_O} + \frac{1}{R} \right)$$

$$\frac{c_{3A} \cdot (p_2 \cdot u_1 - p_1 \cdot u_2)}{V_{ap}} = \left(G_v + K - \frac{D' \cdot R_i}{Z_L} \right) \cdot \left(\frac{1}{D' \cdot R} \right)$$

Simplification leads to:

$$\frac{\hat{v}_{IN}}{\hat{i}_{IN}} = - \frac{\left(\frac{1}{K_m} + \frac{R_i}{Z_L} \right) \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + \left(G_v + K - \frac{D' \cdot R_i}{Z_L} \right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)}{(G_v + K) \cdot \frac{1}{D' \cdot R} - \frac{1}{K_m \cdot Z_O}}$$

Input Impedance:

$$\frac{\hat{v}_{IN}}{\hat{i}_{IN}} = -D'^2 \cdot R \cdot \frac{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O} \right) + (G_v + K) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)}{(G_v + K) \cdot D' - \frac{1}{K_m} \cdot \left(\frac{D'^2 \cdot R}{Z_O} \right)} \quad (\text{B.45})$$

Boost – Control-to-Inductor Current

From equation B.25 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation B.33:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= \hat{v}_C - \hat{v}_O \cdot c_3 \\ p_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) &= c_1 \cdot (\hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (c_1 \cdot p_3 + c_3 \cdot p_1) &= \hat{v}_C \cdot p_1\end{aligned}\tag{B.46}$$

From equations B.26 with $\hat{v}_{IN} = 0$ and equation B.33:

$$\begin{aligned}\hat{d} \cdot u_1 &= \hat{i}_L + \hat{v}_O \cdot u_3 & \hat{d} \cdot c_1 &= \hat{v}_C - \hat{v}_O \cdot c_3 \\ u_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) &= c_1 \cdot (\hat{i}_L + \hat{v}_O \cdot u_3) \\ \hat{v}_O \cdot (c_1 \cdot u_3 + c_3 \cdot u_1) &= \hat{v}_C \cdot u_1 - \hat{i}_L \cdot c_1\end{aligned}\tag{B.47}$$

Equate B.46 to B.47:

$$\begin{aligned}(c_1 \cdot p_3 + c_3 \cdot p_1) \cdot (\hat{v}_C \cdot u_1 - \hat{i}_L \cdot c_1) &= \hat{v}_C \cdot p_1 \cdot (c_1 \cdot u_3 + c_3 \cdot u_1) \\ \frac{\hat{i}_L}{\hat{v}_C} &= \frac{p_3 \cdot u_1 - p_1 \cdot u_3}{c_1 \cdot p_3 + c_3 \cdot p_1}\end{aligned}\tag{B.48}$$

Substitute for variables:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\left(D'^2 + \frac{Z_L}{Z_O}\right) \cdot \left(\frac{1}{Z_L}\right) - D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right) \cdot \left(\frac{D'}{Z_L}\right)}{\left(\frac{1}{K_m} + \frac{R_i}{Z_L}\right) \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + \left(K - \frac{D' \cdot R_i}{Z_L}\right) \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}$$

Control-to-Inductor Current:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{1}{R} + \frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R}\right)}\tag{B.49}$$

Boost – Current Loop

From equation B.25 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation B.35:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot b_1 &= -\hat{i}'_L \cdot b_5 - \hat{v}_O \cdot b_3 \\ p_1 \cdot (-\hat{i}'_L \cdot b_5 - \hat{v}_O \cdot b_3) &= b_1 \cdot (\hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (b_1 \cdot p_3 + b_3 \cdot p_1) &= -\hat{i}'_L \cdot b_5 \cdot p_1\end{aligned}\tag{B.50}$$

From equation B.25 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation B.26 with $\hat{v}_{IN} = 0$:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot u_1 &= \hat{i}_L + \hat{v}_O \cdot u_3 \\ p_1 \cdot (\hat{i}_L + \hat{v}_O \cdot u_3) &= u_1 \cdot (\hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (p_3 \cdot u_1 - p_1 \cdot u_3) &= \hat{i}_L \cdot p_1\end{aligned}\tag{B.51}$$

Equate B.50 to B.51:

$$\begin{aligned}(b_1 \cdot p_3 + b_3 \cdot p_1) \cdot \hat{i}_L \cdot p_1 &= -\hat{i}'_L \cdot b_5 \cdot p_1 \cdot (p_3 \cdot u_1 - p_1 \cdot u_3) \\ \frac{\hat{i}_L}{\hat{i}'_L} &= \frac{-b_5 \cdot (p_3 \cdot u_1 - p_1 \cdot u_3)}{b_1 \cdot p_3 + b_3 \cdot p_1}\end{aligned}\tag{B.52}$$

Substitute for variables:

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\left(D'^2 + \frac{Z_L}{Z_O} \right) \cdot \left(\frac{1}{Z_L} \right) - D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right) \cdot \left(\frac{D'}{Z_L} \right) \right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)}$$

Current Loop ($\hat{v}_C = 0$):

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\frac{1}{R} + \frac{1}{Z_O} \right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + K \cdot D' \cdot \left(1 - \frac{Z_L}{D'^2 \cdot R} \right)}\tag{B.53}$$

Boost – Derivation of Factored Form

Starting with the control-to-output gain of equation B.37, regroup and factor dc terms:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i} \cdot \frac{\left(1 - \frac{Z_L}{D'^2 \cdot R}\right) \cdot \frac{Z_O}{R_O}}{1 + \frac{Z_L}{K_m \cdot R_i} + \left(1 + \frac{D'^2 \cdot R}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) - \frac{K \cdot Z_L}{D' \cdot R_i}\right) \cdot \frac{Z_O}{R}}$$

Expand Z_O and Z_L :

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i} \cdot \frac{\left(1 - \frac{s \cdot L + R_L + R_S}{D'^2 \cdot R}\right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left(1 + \frac{s \cdot L + R_L + R_S}{K_m \cdot R_i}\right) \cdot (1 + s \cdot C_O \cdot (R_O + R_C)) + \left(1 + \frac{R \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) - \frac{K \cdot (s \cdot L + R_L + R_S)}{R_i \cdot D'}\right) \cdot \frac{R_O}{R} \cdot (1 + s \cdot C_O \cdot R_C)}$$

Expand the denominator:

$$1 + s \cdot C_O \cdot (R_O + R_C)$$

$$\frac{R_L + R_S}{K_m \cdot R_i} + s \cdot C_O \cdot \frac{(R_O + R_C) \cdot (R_L + R_S)}{K_m \cdot R_i} + \frac{s \cdot L}{K_m \cdot R_i} + s \cdot L \cdot s \cdot C_O \cdot \frac{R_O + R_C}{K_m \cdot R_i}$$

$$\frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) + s \cdot C_O \cdot R_C \cdot \left(\frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right)\right)$$

$$- \frac{R_O}{R} \cdot \frac{K \cdot (R_L + R_S)}{R_i \cdot D'} - s \cdot C_O \cdot \frac{R_O}{R} \cdot \frac{R_C \cdot K \cdot (R_L + R_S)}{R_i \cdot D'} - s \cdot L \cdot \frac{R_O}{R} \cdot \frac{K}{R_i \cdot D'} - s \cdot L \cdot s \cdot C_O \cdot \frac{R_O}{R} \cdot \frac{R_C \cdot K}{R_i \cdot D'}$$

Group like terms:

$$1 + \frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{1}{K_m} - \frac{R_O}{R} \cdot \frac{K}{D'}\right)$$

$$+ s \cdot C_O \cdot \left(R_O + R_C \cdot \left(1 + \frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) \right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{R_O + R_C}{K_m} - \frac{R_O}{R} \cdot \frac{K \cdot R_C}{D'} \right) \right)$$

$$+ \frac{s \cdot L}{K_m \cdot R_i} \cdot \left(1 - \frac{R_O}{R} \cdot \frac{K \cdot K_m}{D'} + s \cdot C_O \cdot \left(R_O + R_C \cdot \left(1 - \frac{R_O}{R} \cdot \frac{K \cdot K_m}{D'} \right) \right) \right)$$

The transfer function can now be expressed as:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i} \cdot \frac{\left(K_B - \frac{s \cdot L}{D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{K_D + s \cdot C_O \cdot K_C + \left(\frac{s \cdot L}{K_m \cdot R_i} \right) \cdot (K_K + s \cdot C_O \cdot K'_C)} \quad (\text{B.54})$$

Where:

$$K_D = 1 + \frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{1}{K_m} - \frac{R_O}{R} \cdot \frac{K}{D'} \right) \quad (\text{B.55})$$

$$K_C = R_O + R_C \cdot \left(1 + \frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) \right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{R_O + R_C}{K_m} - \frac{R_O}{R} \cdot \frac{K \cdot R_C}{D'} \right)$$

$$K_B = 1 - \frac{R_L + R_S}{D'^2 \cdot R} \quad K_K = 1 - \frac{R_O}{R} \cdot \frac{K \cdot K_m}{D'} \quad K'_C = R_O + R_C \cdot K_K$$

By factoring this becomes:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D' \cdot K_B}{R_i \cdot K_D} \cdot \frac{\left(1 - s \cdot \frac{L}{K_B \cdot D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left[1 + s \cdot C_O \cdot \frac{K_C}{K_D} \right] \cdot \left[1 + \left(\frac{s \cdot L}{K_m \cdot R_i} \right) \cdot \left(\frac{K_K + s \cdot C_O \cdot K'_C}{K_D + s \cdot C_O \cdot K_C} \right) \right]} \quad (\text{B.56})$$

To simplify:

$$\begin{aligned} K_D &\approx 1 + \frac{R_O}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) & K_B &\approx 1 \\ K_C &\approx K'_C \approx R_O & \frac{K_K + s \cdot C_O \cdot K'_C}{K_D + s \cdot C_O \cdot K_C} &\approx 1 \end{aligned} \quad (\text{B.57})$$

The simplified expression is:

$$\frac{\hat{v}_O}{\hat{v}_C} \approx \frac{R_O \cdot D'}{R_i \cdot K_D} \cdot \frac{\left(1 - s \cdot \frac{L}{D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left(1 + s \cdot \frac{C_O \cdot R_O}{K_D} \right) \cdot \left(1 + s \cdot \frac{L}{K_m \cdot R_i} \right)} \quad (\text{B.58})$$

Boost – Linear Equation Results

Results from the linear equations were compared to the linear model transfer functions, showing excellent agreement. No sampling-gain term was used for the comparison. For the output impedance and input impedance, $G_V = 3$ represents the typical mid-band error amplifier gain.

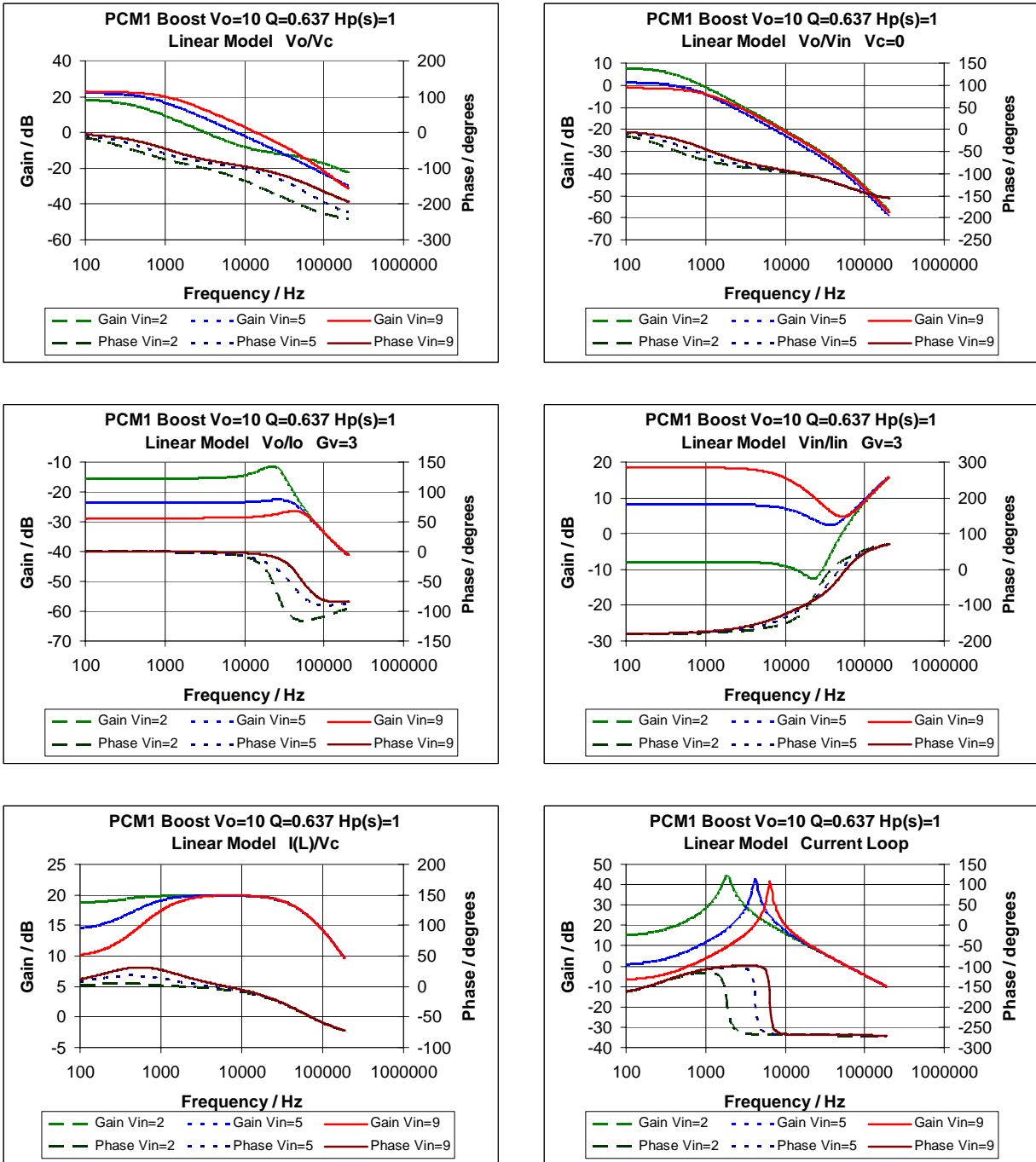


Figure B-4. Boost linear equation results, without sampling-gain term.

Equate C.12 to C.13:

$$\hat{d} \cdot \frac{I_L}{D'} + \hat{i}_O \cdot \frac{1}{D'} = \hat{i}_{IN} \frac{1}{D} - \hat{d} \cdot \frac{I_L}{D}$$

$$\hat{d} \cdot I_L + \hat{i}_O \cdot D = \hat{i}_{IN} \cdot D' \quad (\text{C.14})$$

This allows the input current to be expressed as:

$$\hat{i}_{IN} = \hat{d} \cdot \frac{I_L}{D'} + \hat{i}_O \cdot \frac{D}{D'} \quad (\text{C.15})$$

Duty Cycle to Output Voltage

Equations for voltage in terms of current:

$$\hat{v}_{SW} = \hat{i}_L \cdot Z_L \quad \hat{v}_O = (\hat{i}_O + \hat{i}'_O) \cdot Z_O \quad (\text{C.16}), (\text{C.17})$$

Solving C.17 for output current:

$$\hat{i}_O = \hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O \quad (\text{C.18})$$

From C.12, C.16 and C.18:

$$\hat{v}_{SW} = \hat{d} \cdot \frac{I_L \cdot Z_L}{D'} + \hat{v}_O \cdot \frac{Z_L}{D' \cdot Z_O} - \hat{i}'_O \cdot \frac{Z_L}{D'} \quad (\text{C.19})$$

Write the equation for the switch voltage in terms of circuit voltages:

$$\hat{v}_{SW} = -\hat{v}_O + \left(\hat{v}_{IN} + \frac{V_{ap}}{D} \cdot \hat{d} - (-\hat{v}_O) \right) \cdot D$$

$$\hat{v}_{SW} = \hat{v}_{IN} \cdot D + \hat{d} \cdot V_{ap} - \hat{v}_O \cdot D' \quad (\text{C.20})$$

Equate C.20 to C.19:

$$\hat{v}_{IN} \cdot D + \hat{d} \cdot V_{ap} - \hat{v}_O \cdot D' = \hat{d} \cdot \frac{I_L \cdot Z_L}{D'} + \hat{v}_O \cdot \frac{Z_L}{D' \cdot Z_O} - \hat{i}'_O \cdot \frac{Z_L}{D'}$$

$$\hat{d} \cdot \left(V_{ap} - \frac{I_L \cdot Z_L}{D'} \right) = -\hat{v}_{IN} \cdot D + \hat{v}_O \cdot \left(D' + \frac{Z_L}{D' \cdot Z_O} \right) - \hat{i}'_O \cdot \frac{Z_L}{D'} \quad (\text{C.21})$$

Substituting for I_L and arranging terms:

$$\hat{d} \cdot V_{ap} \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right) = -\hat{v}_{IN} \cdot D \cdot D' + \hat{v}_O \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) - \hat{i}'_O \cdot Z_L \quad (\text{C.22})$$

From equation C.22 with $\hat{i}'_O = 0$:

$$\hat{v}_O = \left(\hat{v}_{IN} \cdot \frac{D}{D'} + \hat{d} \cdot \frac{V_{ap}}{D'} \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right) \right) \cdot \left(\frac{Z_O}{Z_O + \frac{Z_L}{D'^2}} \right) \quad (\text{C.23})$$

Equations C.15 and C.23 define the unified power stage of reference [3]:

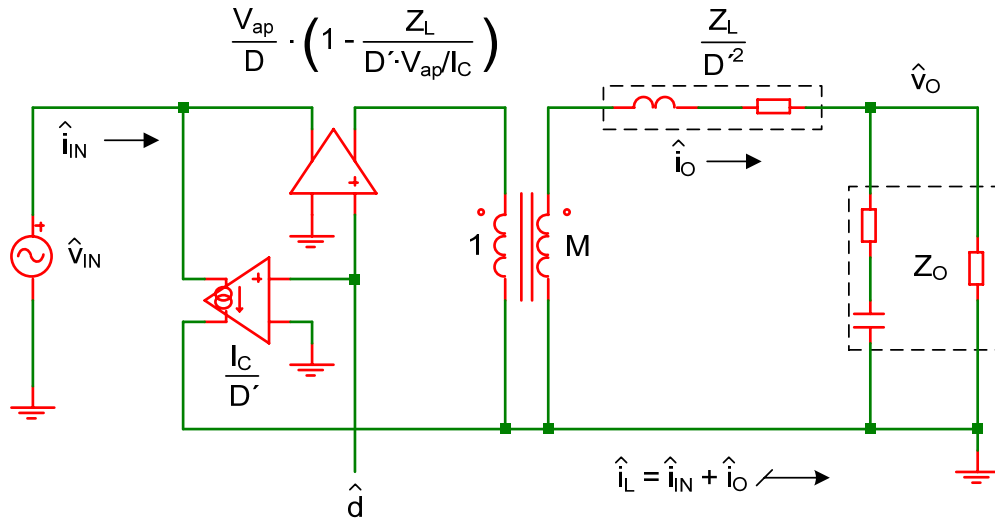


Figure C-2. Buck-boost unified power-stage model.

Duty Cycle to Input Current and Output Voltage

From C.15:

$$\hat{d} \cdot \frac{I_L}{D'} = \hat{i}_{IN} - \hat{i}_O \frac{D}{D'} \quad (\text{C.24})$$

Since $\hat{i}_O = \hat{v}_O \cdot \frac{1}{Z_O} - \hat{i}'_O$ from C.18, and substituting for I_L :

$$\hat{d} \cdot V_{ap} \cdot \frac{D}{D'^2 \cdot R} = \hat{i}_{IN} - \hat{v}_O \cdot \frac{D}{D' \cdot Z_O} + \hat{i}'_O \cdot \frac{D}{D'} \quad (\text{C.25})$$

Duty Cycle to Inductor Current and Output Voltage

From C.12:

$$\hat{i}_O = \hat{i}_L \cdot D' - \hat{d} \cdot I_L \quad (\text{C.26})$$

From C.17 and C.26:

$$\hat{v}_O = (\hat{i}_L \cdot D' - \hat{d} \cdot I_L + \hat{i}'_O) \cdot Z_O \quad (\text{C.27})$$

From C.27:

$$\hat{d} \cdot \frac{I_L}{D'} = \hat{i}_L - \hat{v}_O \cdot \frac{1}{D' \cdot Z_O} + \hat{i}'_O \cdot \frac{1}{D'} \quad (\text{C.28})$$

Substituting for I_L :

$$\hat{d} \cdot V_{ap} \cdot \frac{D}{D'^2 \cdot R} = \hat{i}_L - \hat{v}_O \cdot \frac{1}{D' \cdot Z_O} + \hat{i}'_O \cdot \frac{1}{D'} \quad (\text{C.29})$$

Buck-Boost Power Stage Summary

Equations C.22, C.25 and C.29 from the power stage allow derivation of all transfer functions.

Use equation C.22 for all transfer functions. Let:

$$p_1 = V_{ap} \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right) \quad p_2 = D \cdot D' \quad p_3 = D'^2 + \frac{Z_L}{Z_O} \quad p_4 = Z_L$$

$$\hat{d} \cdot p_1 = -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4 \quad (\text{C.30})$$

Use equation C.25 for input impedance. Let $\hat{i}'_O = 0$:

$$q_1 = V_{ap} \cdot \frac{D}{D'^2 \cdot R} \quad q_3 = \frac{D}{D' \cdot Z_O}$$

$$\hat{d} \cdot q_1 = \hat{i}_{IN} - \hat{v}_O \cdot q_3 \quad (\text{C.31})$$

Use equation C.29 for control-to-inductor current and current loop. Let $\hat{i}'_O = 0$:

$$u_1 = V_{ap} \cdot \frac{D}{D'^2 \cdot R} \quad u_3 = \frac{1}{D' \cdot Z_O}$$

$$\hat{d} \cdot u_1 = \hat{i}_L - \hat{v}_O \cdot u_3 \quad (\text{C.32})$$

Current-Mode Buck-Boost – Control Stage

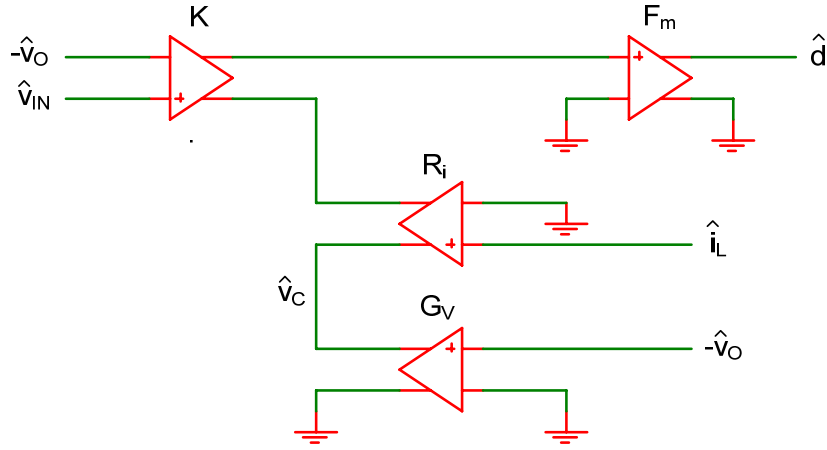


Figure C-3. Buck-boost control-stage model.

Write the transfer function from the control-stage model:

$$\hat{d} = F_m \cdot (\hat{v}_C - \hat{i}_L \cdot R_i - (\hat{v}_{IN} + \hat{v}_O) \cdot K) \quad (\text{C.33})$$

Relate inductor current to output voltage. From power-stage equation C.28:

$$\hat{i}_L = \hat{d} \cdot \frac{I_L}{D'} + \hat{v}_O \cdot \frac{1}{D' \cdot Z_O} - \hat{i}'_O \cdot \frac{1}{D'} \quad (\text{C.34})$$

Substitute C.34 into C.33 to eliminate the inductor current term:

$$\begin{aligned} \hat{d} &= F_m \cdot \left(\hat{v}_C - \left(\hat{d} \cdot \frac{I_L}{D'} + \hat{v}_O \cdot \frac{1}{D' \cdot Z_O} - \hat{i}'_O \cdot \frac{1}{D'} \right) \cdot R_i - (\hat{v}_{IN} + \hat{v}_O) \cdot K \right) \\ \hat{d} \cdot \left(\frac{1}{F_m} + \frac{I_L \cdot R_i}{D'} \right) &= \hat{v}_C - \hat{v}_{IN} \cdot K - \hat{v}_O \cdot \left(\frac{R_i}{D' \cdot Z_O} + K \right) + \hat{i}'_O \cdot \frac{R_i}{D'} \end{aligned} \quad (\text{C.35})$$

Substituting for I_L :

$$\hat{d} \cdot V_{ap} \cdot \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) = \hat{v}_C - \hat{v}_{IN} \cdot K - \hat{v}_O \cdot \left(\frac{R_i}{D' \cdot Z_O} + K \right) + \hat{i}'_O \cdot \frac{R_i}{D'} \quad (\text{C.36})$$

For closed voltage-loop equations, let $\hat{v}_C = -G_V \cdot \hat{v}_O$. From equation C.36:

$$\hat{d} \cdot V_{ap} \cdot \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) = -\hat{v}_{IN} \cdot K - \hat{v}_O \cdot \left(\frac{R_i}{D' \cdot Z_O} + G_V + K \right) + \hat{i}'_O \cdot \frac{R_i}{D'} \quad (\text{C.37})$$

General expression from equation C.33:

$$\hat{d} \cdot V_{ap} \cdot \frac{1}{K_m} = \hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{IN} \cdot K - \hat{v}_O \cdot K \quad (\text{C.38})$$

Buck-Boost Control Stage Summary

Equations C.36, C.37 and C.38 from the control stage allow derivation of all transfer functions.

Use equation C.36 for control-to-output and control-to-inductor current. Let $\hat{v}_{IN} = 0$ and $\hat{i}'_O = 0$:

$$c_1 = V_{ap} \cdot \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) \quad c_3 = \frac{R_i}{D' \cdot Z_O} + K$$

$$\hat{d} \cdot c_1 = \hat{v}_C - \hat{v}_O \cdot c_3 \quad (\text{C.39})$$

Use equation C.37 for line-to-output, output impedance and input impedance. Let:

$$c_1 = V_{ap} \cdot \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) \quad c_2 = K \quad c_{3A} = \frac{R_i}{D' \cdot Z_O} + G_V + K \quad c_4 = \frac{R_i}{D'}$$

$$\hat{d} \cdot c_1 = -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} + \hat{i}'_O \cdot c_4 \quad (\text{C.40})$$

Use equation C.38 for the current loop with $\hat{v}_C = 0$ and $\hat{v}_{IN} = 0$. Designate \hat{i}'_L as the control input:

$$b_1 = V_{ap} \cdot \frac{1}{K_m} \quad b_5 = R_i \quad b_3 = K$$

$$\hat{d} \cdot b_1 = -\hat{i}'_L \cdot b_5 - \hat{v}_O \cdot b_3 \quad (\text{C.41})$$

Buck-Boost – Control-to-Output

From equation C.30 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation C.39:

$$\hat{d} \cdot p_1 = \hat{v}_O \cdot p_3 \quad \hat{d} \cdot c_1 = \hat{v}_C - \hat{v}_O \cdot c_3$$

$$p_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) = c_1 \cdot (\hat{v}_O \cdot p_3)$$

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{p_1}{c_1 \cdot p_3 + c_3 \cdot p_1} \quad (\text{C.42})$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R}\right) \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + \left(\frac{R_i}{D' \cdot Z_O} + K\right) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Control-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)} \quad (\text{C.43})$$

Buck-Boost – Line-to-Output

From equations C.30 and C.40 with $\hat{i}'_O = 0$:

$$\begin{aligned}\hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} \\ p_1 \cdot (-\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A}) &= c_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \frac{\hat{v}_O}{\hat{v}_{IN}} &= \frac{c_1 \cdot p_2 - c_2 \cdot p_1}{c_1 \cdot p_3 + c_{3A} \cdot p_1}\end{aligned}\tag{C.44}$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{\left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R}\right) \cdot D \cdot D' - K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R}\right) \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + \left(\frac{R_i}{D' \cdot Z_O} + G_V + K\right) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Line-to-Output:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{D \cdot D' \cdot \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R}\right) - K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}\tag{C.45}$$

Buck-Boost – Output Impedance

From equations C.30 and C.40 with $\hat{v}_{IN} = 0$:

$$\begin{aligned} \hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4 & \hat{d} \cdot c_1 &= -\hat{v}_O \cdot c_{3A} + \hat{i}'_O \cdot c_4 \\ p_1 \cdot (-\hat{v}_O \cdot c_{3A} + \hat{i}'_O \cdot c_4) &= c_1 \cdot (\hat{v}_O \cdot p_3 - \hat{i}'_O \cdot p_4) \\ \frac{\hat{v}_O}{\hat{i}'_O} &= \frac{c_1 \cdot p_4 + c_4 \cdot p_1}{c_1 \cdot p_3 + c_{3A} \cdot p_1} \end{aligned} \quad (C.46)$$

Substitute for variables:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) \cdot Z_L + \left(\frac{R_i}{D'} \right) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)}{\left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + \left(\frac{R_i}{D' \cdot Z_O} + G_V + K \right) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)}$$

Output Impedance:

$$\frac{\hat{v}_O}{\hat{i}'_O} = \frac{\frac{1}{K_m} \cdot Z_L + R_i}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O} \right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)} \quad (C.47)$$

Buck-Boost – Input Impedance

From equations C.30 and C.40 with $\hat{i}'_O = 0$:

$$\begin{aligned}\hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= -\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A} \\ p_1 \cdot (-\hat{v}_{IN} \cdot c_2 - \hat{v}_O \cdot c_{3A}) &= c_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (c_1 \cdot p_3 + c_{3A} \cdot p_1) &= \hat{v}_{IN} \cdot (c_1 \cdot p_2 - c_2 \cdot p_1)\end{aligned}\tag{C.48}$$

From equation C.30 with $\hat{i}'_O = 0$ and equation C.31:

$$\begin{aligned}\hat{d} \cdot p_1 &= -\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3 & \hat{d} \cdot q_1 &= \hat{i}_{IN} - \hat{v}_O \cdot q_3 \\ p_1 \cdot (\hat{i}_{IN} - \hat{v}_O \cdot q_3) &= q_1 \cdot (-\hat{v}_{IN} \cdot p_2 + \hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (p_3 \cdot q_1 + p_1 \cdot q_3) &= \hat{v}_{IN} \cdot p_2 \cdot q_1 + \hat{i}_{IN} \cdot p_1\end{aligned}\tag{C.49}$$

Equate C.48 to C.49:

$$\begin{aligned}(\hat{v}_{IN} \cdot p_2 \cdot q_1 + \hat{i}_{IN} \cdot p_1) \cdot (c_1 \cdot p_3 + c_{3A} \cdot p_1) &= \hat{v}_{IN} \cdot (c_1 \cdot p_2 - c_2 \cdot p_1) \cdot (p_3 \cdot q_1 + p_1 \cdot q_3) \\ \frac{\hat{v}_{IN}}{\hat{i}_{IN}} &= -\frac{c_1 \cdot p_3 + c_{3A} \cdot p_1}{c_2 \cdot p_3 \cdot q_1 + c_{3A} \cdot p_2 \cdot q_1 + c_2 \cdot p_1 \cdot q_3 - c_1 \cdot p_2 \cdot q_3} \\ \frac{\hat{v}_{IN}}{\hat{i}_{IN}} &= -\frac{c_1 \cdot p_3 + c_{3A} \cdot p_1}{c_2 \cdot (p_3 \cdot q_1 + p_1 \cdot q_3) + p_2 \cdot (c_{3A} \cdot q_1 - c_1 \cdot q_3)}\end{aligned}\tag{C.50}$$

Substitute for variables:

Numerator:

$$\frac{c_1 \cdot p_3 + c_{3A} \cdot p_1}{V_{ap}} = \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + \left(\frac{R_i}{D' \cdot Z_O} + G_V + K \right) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)$$

Buck-Boost – Input Impedance (continued)

Denominator:

$$\frac{c_2 \cdot (p_3 \cdot q_1 + p_1 \cdot q_3)}{V_{ap}} = K \cdot \left(\left(D'^2 + \frac{Z_L}{Z_O} \right) \cdot \frac{D}{D'^2 \cdot R} + D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right) \cdot \frac{D}{D' \cdot Z_O} \right)$$

$$\frac{p_2 \cdot (c_{3A} \cdot q_1 - c_1 \cdot q_3)}{V_{ap}} = D \cdot D' \cdot \left(\left(\frac{R_i}{D' \cdot Z_O} + G_V + K \right) \cdot \frac{D}{D'^2 \cdot R} - \left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R} \right) \cdot \frac{D}{D' \cdot Z_O} \right)$$

$$\frac{c_2 \cdot (p_3 \cdot q_1 + p_1 \cdot q_3)}{V_{ap}} = \frac{K \cdot D}{D'^2 \cdot R} \cdot \left(D'^2 + \frac{D'^2 \cdot R + D' \cdot Z_L}{Z_O} \right)$$

$$\frac{p_2 \cdot (c_{3A} \cdot q_1 - c_1 \cdot q_3)}{V_{ap}} = \frac{D^2}{D'^2 \cdot R} \cdot \left(\frac{R_i \cdot D'}{Z_O} + (G_V + K) \cdot D' - \frac{1}{K_m} \cdot \frac{D'^2 \cdot R}{Z_O} \right)$$

Input Impedance:

$$\frac{\hat{v}_{IN}}{\hat{i}_{IN}} = - \frac{D'^2 \cdot R}{D^2} \cdot \frac{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O} \right) + (G_V + K) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)}{\frac{K}{D} \cdot \left(D'^2 + \frac{D'^2 \cdot R + D' \cdot Z_L}{Z_O} \right) + R_i \cdot \left(\frac{D'}{Z_O} \right) + (G_V + K) \cdot D' - \frac{1}{K_m} \cdot \left(\frac{D'^2 \cdot R}{Z_O} \right)} \quad (\text{C.51})$$

Buck-Boost – Control-to-Inductor Current

From equation C.30 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation C.39:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot c_1 &= \hat{v}_C - \hat{v}_O \cdot c_3 \\ p_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) &= c_1 \cdot (\hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (c_1 \cdot p_3 + c_3 \cdot p_1) &= \hat{v}_C \cdot p_1\end{aligned}\tag{C.52}$$

From equations C.32 and C.39:

$$\begin{aligned}\hat{d} \cdot u_1 &= \hat{i}_L - \hat{v}_O \cdot u_3 & \hat{d} \cdot c_1 &= \hat{v}_C - \hat{v}_O \cdot c_3 \\ u_1 \cdot (\hat{v}_C - \hat{v}_O \cdot c_3) &= c_1 \cdot (\hat{i}_L - \hat{v}_O \cdot u_3) \\ \hat{v}_O \cdot (c_1 \cdot u_3 - c_3 \cdot u_1) &= \hat{i}_L \cdot c_1 - \hat{v}_C \cdot u_1\end{aligned}\tag{C.53}$$

Equate C.52 to C.53:

$$\begin{aligned}(c_1 \cdot p_3 + c_3 \cdot p_1) \cdot (\hat{i}_L \cdot c_1 - \hat{v}_C \cdot u_1) &= \hat{v}_C \cdot p_1 \cdot (c_1 \cdot u_3 - c_3 \cdot u_1) \\ \frac{\hat{i}_L}{\hat{v}_C} &= \frac{p_1 \cdot u_3 + p_3 \cdot u_1}{c_1 \cdot p_3 + c_3 \cdot p_1}\end{aligned}\tag{C.54}$$

Substitute for variables:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right) \cdot \frac{1}{D' \cdot Z_O} + \left(D'^2 + \frac{Z_L}{Z_O}\right) \cdot \frac{D}{D'^2 \cdot R}}{\left(\frac{1}{K_m} + \frac{D \cdot R_i}{D'^2 \cdot R}\right) \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + \left(\frac{R_i}{D' \cdot Z_O} + K\right) \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}$$

Control-to-Inductor Current:

$$\frac{\hat{i}_L}{\hat{v}_C} = \frac{\frac{D}{R} + \frac{1}{Z_O}}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O}\right) + R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O}\right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right)}\tag{C.55}$$

Buck-Boost – Current Loop

From equation C.30 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation C.41:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot b_1 &= -\hat{i}'_L \cdot b_5 - \hat{v}_O \cdot b_3 \\ p_1 \cdot (-\hat{i}'_L \cdot b_5 - \hat{v}_O \cdot b_3) &= b_1 \cdot (\hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (b_1 \cdot p_3 + b_3 \cdot p_1) &= -\hat{i}'_L \cdot b_5 \cdot p_1\end{aligned}\tag{C.56}$$

From equation C.30 with $\hat{v}_{IN} = 0$, $\hat{i}'_O = 0$ and equation C.32:

$$\begin{aligned}\hat{d} \cdot p_1 &= \hat{v}_O \cdot p_3 & \hat{d} \cdot u_1 &= \hat{i}_L - \hat{v}_O \cdot u_3 \\ p_1 \cdot (\hat{i}_L - \hat{v}_O \cdot u_3) &= u_1 \cdot (\hat{v}_O \cdot p_3) \\ \hat{v}_O \cdot (p_3 \cdot u_1 + p_1 \cdot u_3) &= \hat{i}_L \cdot p_1\end{aligned}\tag{C.57}$$

Equate C.56 to C.57:

$$\begin{aligned}(b_1 \cdot p_3 + b_3 \cdot p_1) \cdot \hat{i}_L \cdot p_1 &= -\hat{i}'_L \cdot b_5 \cdot p_1 \cdot (p_3 \cdot u_1 + p_1 \cdot u_3) \\ \frac{\hat{i}_L}{\hat{i}'_L} &= \frac{-b_5 \cdot (p_1 \cdot u_3 + p_3 \cdot u_1)}{b_1 \cdot p_3 + b_3 \cdot p_1}\end{aligned}\tag{C.58}$$

Substitute for variables:

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right) \frac{1}{D' \cdot Z_O} + \left(D'^2 + \frac{Z_L}{Z_O} \right) \cdot \frac{D}{D'^2 \cdot R} \right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)}$$

Current Loop ($\hat{v}_C = 0$):

$$\frac{\hat{i}_L}{\hat{i}'_L} = \frac{-R_i \cdot \left(\frac{D}{R} + \frac{1}{Z_O} \right)}{\frac{1}{K_m} \cdot \left(D'^2 + \frac{Z_L}{Z_O} \right) + K \cdot D' \cdot \left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R} \right)}\tag{C.59}$$

Buck-Boost – Derivation of Factored Form

Starting with the control-to-output gain of equation C.43, regroup and factor the dc terms:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i} \cdot \frac{\left(1 - \frac{D \cdot Z_L}{D'^2 \cdot R}\right) \cdot \frac{Z_O}{R_O}}{1 + \frac{Z_L}{K_m \cdot R_i} + \left(D + \frac{D'^2 \cdot R}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) - \frac{K \cdot D \cdot Z_L}{D' \cdot R_i}\right) \cdot \frac{Z_O}{R}}$$

Expand Z_O and Z_L :

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i} \cdot \frac{\left(1 - \frac{D \cdot (s \cdot L + R_L + R_S)}{D'^2 \cdot R}\right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left(1 + \frac{s \cdot L + R_L + R_S}{K_m \cdot R_i}\right) \cdot (1 + s \cdot C_O \cdot (R_O + R_C)) + \left(D + \frac{R \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) - \frac{K \cdot D \cdot (s \cdot L + R_L + R_S)}{R_i \cdot D'}\right) \cdot \frac{R_O}{R} \cdot (1 + s \cdot C_O \cdot R_C)}$$

Expand the denominator:

$$\begin{aligned} & 1 + s \cdot C_O \cdot (R_O + R_C) \\ & \frac{R_L + R_S}{K_m \cdot R_i} + s \cdot C_O \cdot \frac{(R_O + R_C) \cdot (R_L + R_S)}{K_m \cdot R_i} + \frac{s \cdot L}{K_m \cdot R_i} + s \cdot L \cdot s \cdot C_O \cdot \frac{R_O + R_C}{K_m \cdot R_i} \\ & \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) + s \cdot C_O \cdot R_C \cdot \left(\frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right)\right) \\ & - \frac{R_O}{R} \cdot \frac{K \cdot D \cdot (R_L + R_S)}{R_i \cdot D'} - s \cdot C_O \cdot \frac{R_O}{R} \cdot \frac{R_C \cdot K \cdot D \cdot (R_L + R_S)}{R_i \cdot D'} - s \cdot L \cdot \frac{R_O}{R} \cdot \frac{K \cdot D}{R_i \cdot D'} - s \cdot L \cdot s \cdot C_O \cdot \frac{R_O}{R} \cdot \frac{R_C \cdot K \cdot D}{R_i \cdot D'} \end{aligned}$$

Group like terms:

$$\begin{aligned} & 1 + \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{1}{K_m} - \frac{R_O}{R} \cdot \frac{K \cdot D}{D'}\right) \\ & s \cdot C_O \cdot \left(R_O + R_C \cdot \left(1 + \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'}\right)\right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{R_O + R_C}{K_m} - \frac{R_O}{R} \cdot \frac{K \cdot D \cdot R_C}{D'}\right)\right) \\ & \frac{s \cdot L}{K_m \cdot R_i} \cdot \left(1 - \frac{R_O}{R} \cdot \frac{K \cdot D \cdot K_m}{D'} + s \cdot C_O \cdot \left(R_O + R_C \cdot \left(1 - \frac{R_O}{R} \cdot \frac{K \cdot D \cdot K_m}{D'}\right)\right)\right) \end{aligned}$$

The transfer function can now be expressed as:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D'}{R_i} \cdot \frac{\left(K_B - \frac{s \cdot D \cdot L}{D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{K_D + s \cdot C_O \cdot K_C + \left(\frac{s \cdot L}{K_m \cdot R_i} \right) \cdot (K_K + s \cdot C_O \cdot K'_C)} \quad (\text{C.60})$$

Where:

$$K_D = 1 + \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{1}{K_m} - \frac{R_O}{R} \cdot \frac{K \cdot D}{D'} \right) \quad (\text{C.61})$$

$$K_C = R_O + R_C \cdot \left(1 + \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) \right) + \frac{R_L + R_S}{R_i} \cdot \left(\frac{R_O + R_C}{K_m} - \frac{R_O}{R} \cdot \frac{K \cdot D \cdot R_C}{D'} \right)$$

$$K_B = 1 - \frac{D \cdot (R_L + R_S)}{D'^2 \cdot R} \quad K_K = 1 - \frac{R_O}{R} \cdot \frac{K \cdot D \cdot K_m}{D'} \quad K'_C = R_O + R_C \cdot K_K$$

By factoring this becomes:

$$\frac{\hat{v}_O}{\hat{v}_C} = \frac{R_O \cdot D' \cdot K_B}{R_i \cdot K_D} \cdot \frac{\left(1 - s \cdot \frac{D \cdot L}{K_B \cdot D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left[1 + s \cdot C_O \cdot \frac{K_C}{K_D} \right] \cdot \left[1 + \left(\frac{s \cdot L}{K_m \cdot R_i} \right) \cdot \left(\frac{K_K + s \cdot C_O \cdot K'_C}{K_D + s \cdot C_O \cdot K_C} \right) \right]} \quad (\text{C.62})$$

To simplify:

$$K_D \approx 1 + \frac{R_O \cdot D}{R} + \frac{R_O \cdot D'^2}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) \quad K_B \approx 1$$

$$K_C \approx K'_C \approx R_O \quad \frac{K_K + s \cdot C_O \cdot K'_C}{K_D + s \cdot C_O \cdot K_C} \approx 1 \quad (\text{C.63})$$

The simplified expression is:

$$\frac{\hat{v}_O}{\hat{v}_C} \approx \frac{R_O \cdot D'}{R_i \cdot K_D} \cdot \frac{\left(1 - s \cdot \frac{D \cdot L}{D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left(1 + s \cdot \frac{C_O \cdot R_O}{K_D} \right) \cdot \left(1 + s \cdot \frac{L}{K_m \cdot R_i} \right)} \quad (\text{C.64})$$

Starting with the line-to-output gain of equation C.45, let $G_v = 0$ for the audio susceptibility transfer function. Regroup and factor the dc terms:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} = \frac{R_O \cdot D \cdot D'}{R_i} \cdot \frac{\left(\frac{1}{K_m} - \frac{K}{D} + \frac{D \cdot R_i}{D'^2 \cdot R} + \frac{K \cdot Z_L}{D'^2 \cdot R} \right) \cdot \frac{Z_O}{R_O}}{1 + \frac{Z_L}{K_m \cdot R_i} + \left(D + \frac{D'^2 \cdot R}{R_i} \cdot \left(\frac{1}{K_m} + \frac{K}{D'} \right) - \frac{K \cdot D \cdot Z_L}{D' \cdot R_i} \right) \cdot \frac{Z_O}{R}}$$

Expanding Z_L and factoring the numerator term results in:

$$\left(\frac{1}{K_m} - \frac{K}{D} + \frac{D \cdot R_i}{D'^2 \cdot R} + \frac{K \cdot (s \cdot L + R_L + R_S)}{D'^2 \cdot R} \right) = K_n \cdot \left(1 + s \cdot \frac{K \cdot L}{K_n \cdot D'^2 \cdot R} \right) \quad (\text{C.65})$$

Where:
$$K_n = \frac{1}{K_m} - \frac{K}{D} + \frac{D \cdot R_i}{D'^2 \cdot R} + \frac{K \cdot (R_L + R_S)}{D'^2 \cdot R} \quad (\text{C.66})$$

To simplify:
$$K_n \approx \frac{1}{K_m} - \frac{K}{D} + \frac{D \cdot R_i}{D'^2 \cdot R} \quad (\text{C.67})$$

Combining with the control-to-output result, the simplified expression is:

$$\frac{\hat{v}_O}{\hat{v}_{IN}} \approx \frac{R_O \cdot D \cdot D' \cdot K_n}{R_i \cdot K_D} \cdot \frac{\left(1 + s \cdot \frac{K \cdot L}{K_n \cdot D'^2 \cdot R} \right) \cdot (1 + s \cdot C_O \cdot R_C)}{\left(1 + s \cdot \frac{C_O \cdot R_O}{K_D} \right) \cdot \left(1 + s \cdot \frac{L}{K_m \cdot R_i} \right)} \quad (\text{C.68})$$

Buck-Boost – Linear Equation Results

Results from the linear equations were compared to the linear model transfer functions, showing excellent agreement. No sampling-gain term was used for the comparison. For the output impedance and input impedance, $G_V = 3$ represents the typical mid-band error amplifier gain.

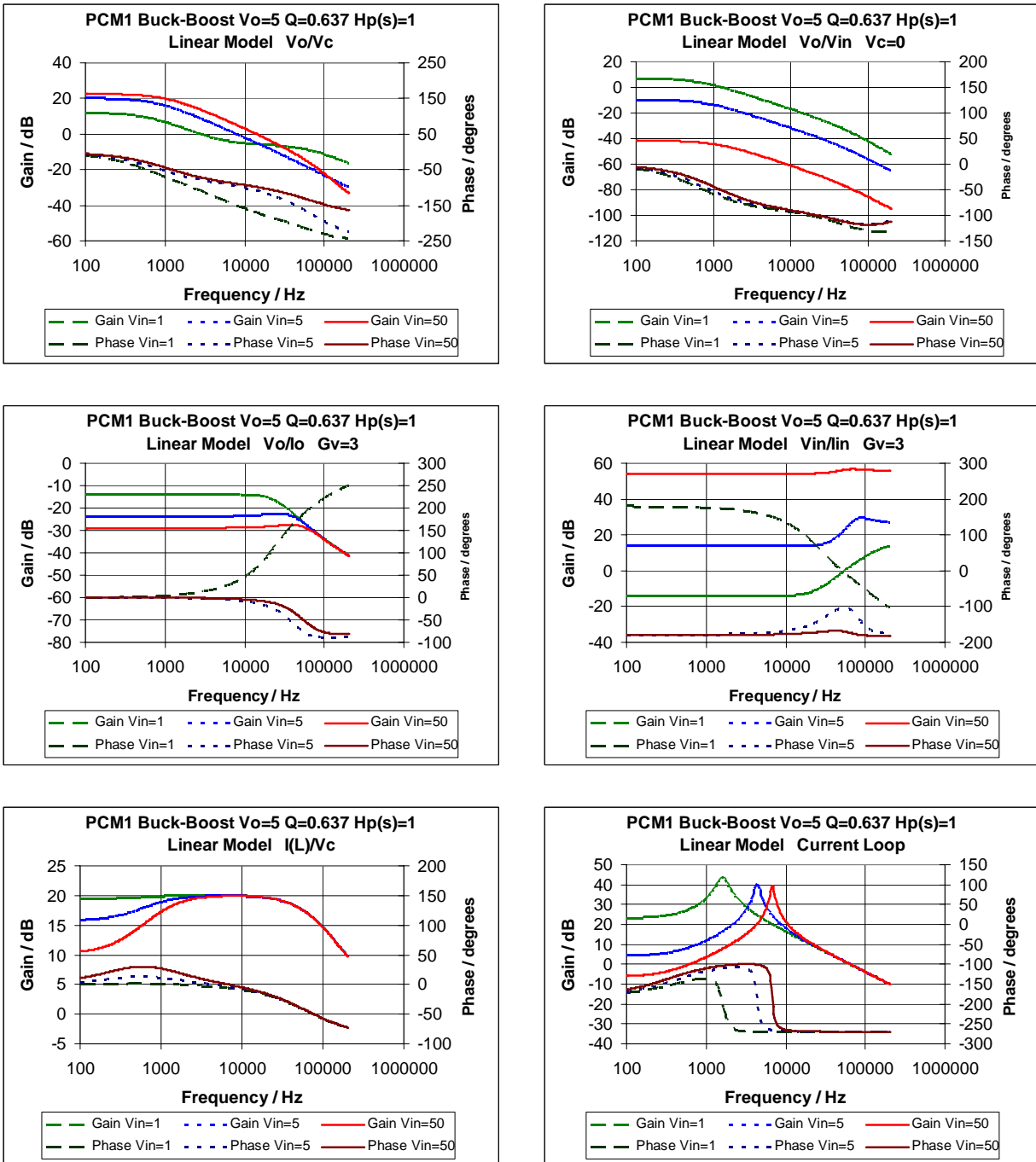


Figure C-4. Buck-boost linear equation results, without sampling-gain term.

Appendix D – Unified Modulator Derivations

These derivations are based on the model from reference [3]. General variables are represented by lower case letters. Perturbed variables are represented by lower case letters with a circumflex. Capital letters denote dc quantities, either fixed or variable. For perturbation and small-signal linearization, products of perturbed variables are ignored. To show the ac small-signal equations, products of dc quantities are not used. As an example:

$$a \cdot x = (\hat{a} + A) \cdot (\hat{x} + X) = \hat{a} \cdot \hat{x} + \hat{a} \cdot X + A \cdot \hat{x} + A \cdot X \quad (D.1)$$

Since the products of perturbed variables are ignored, this leaves two equations:

$$a \cdot x(\text{ac}) = \hat{a} \cdot X + A \cdot \hat{x} \quad a \cdot x(\text{dc}) = A \cdot X \quad (D.2)$$

For the switching regulator duty cycle, $\hat{d}' = 1 - \hat{d}$. After perturbation and linearization, the 1 can be ignored, so $\hat{d}' = -\hat{d}$.

For each topology, the terminal voltage is specified as:

$$\text{Buck} \quad v_{\text{ap}} = v_{\text{IN}} \quad \hat{v}_{\text{ap}} = \hat{v}_{\text{IN}} \quad V_{\text{ap}} = V_{\text{IN}} \quad (D.3)$$

$$\text{Boost} \quad v_{\text{ap}} = v_{\text{O}} \quad \hat{v}_{\text{ap}} = \hat{v}_{\text{O}} \quad V_{\text{ap}} = V_{\text{O}} \quad (D.4)$$

$$\text{Buck-boost} \quad v_{\text{ap}} = v_{\text{IN}} + v_{\text{O}} \quad \hat{v}_{\text{ap}} = \hat{v}_{\text{IN}} + \hat{v}_{\text{O}} \quad V_{\text{ap}} = V_{\text{IN}} + V_{\text{O}} \quad (D.5)$$

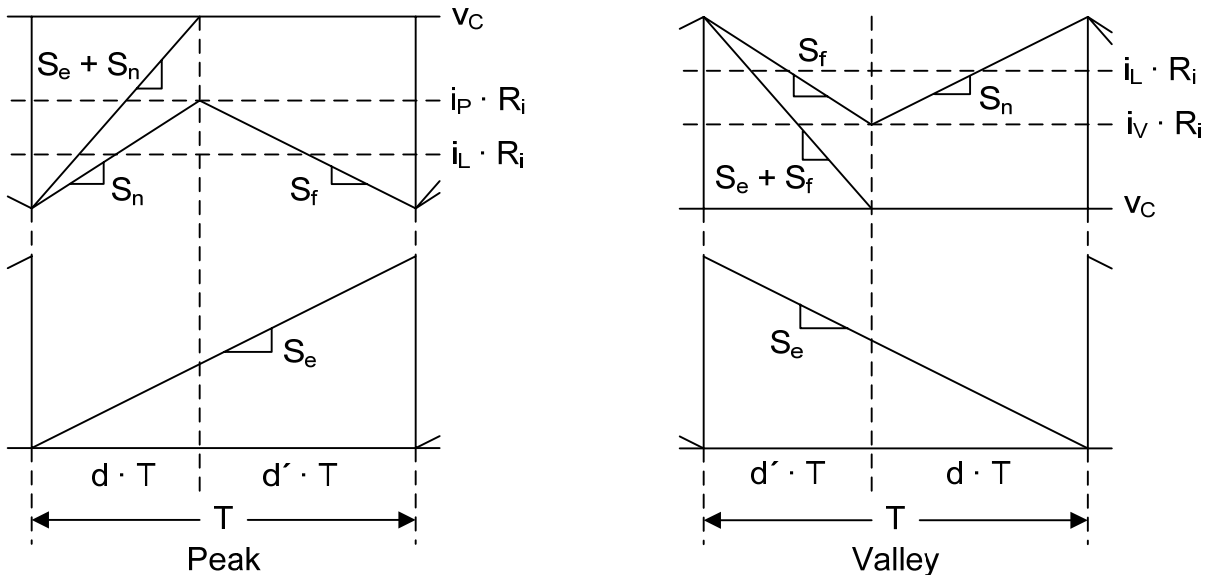


Figure D-1. Peak and valley inductor current, external ramp and control voltage geometry.

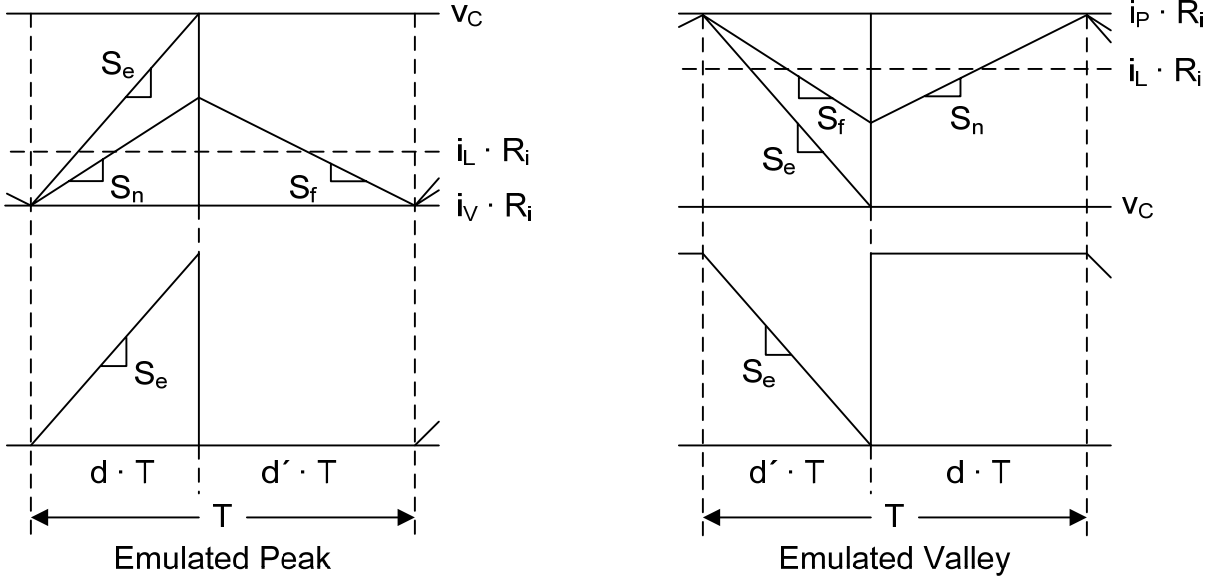


Figure D-2. Emulated peak and valley inductor current, external ramp and control voltage geometry.

From reference [3] with respect to the control voltage, the relationship of the average current to the peak current is described by two equations:

$$i_P \cdot R_i = i_L \cdot R_i + 0.5 \cdot S_n \cdot d \cdot T \quad i_P \cdot R_i = i_L \cdot R_i + 0.5 \cdot S_f \cdot d' \cdot T \quad (\text{D.6})$$

The slopes are described by:

$$S_n = d' \cdot v_{ap} \cdot \frac{R_i}{L} \quad S_f = d \cdot v_{ap} \cdot \frac{R_i}{L} \quad (\text{D.7})$$

Substitution of D.7 into either expression in D.6 leads to the same result:

$$i_P \cdot R_i = i_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} \quad (\text{D.8})$$

Likewise for the valley current:

$$i_V \cdot R_i = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} \quad (\text{D.9})$$

Equations D.8 and D.9 form the basis of the unified modulator, and the derivations which follow. Define V_{SL} as the fixed slope-compensation term and K_{SL} as the proportional slope-compensation term.

$$\text{Mode 1: } V_{SL} = S_e \cdot T \quad \text{Mode 2: } K_{SL} = R_i \cdot \frac{T}{L} \quad (\text{D.10})$$

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Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + V_{SL} \cdot d \quad (D.11)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + V_{SL} \cdot \hat{d} \quad (D.12)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \quad (D.13)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \right) \quad (D.14)$$

PCM2

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + K_{SL} \cdot v_{ap} \cdot d \cdot d \quad (D.15)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + K_{SL} \cdot (\hat{v}_{ap} \cdot D^2 + 2 \cdot V_{ap} \cdot \hat{d} \cdot D) \quad (D.16)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + 2 \cdot K_{SL} \cdot D \right) + \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D^2 \right) \quad (D.17)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + 2 \cdot K_{SL} \cdot D} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D^2 \right) \right) \quad (D.18)$$

VCM1

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} - V_{SL} \cdot d' \quad (\text{D.19})$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) - V_{SL} \cdot \hat{d}' \quad (\text{D.20})$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \quad (\text{D.21})$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \right) \quad (\text{D.22})$$

VCM2

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} - K_{SL} \cdot v_{ap} \cdot d' \cdot d' \quad (\text{D.23})$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) - K_{SL} \cdot (\hat{v}_{ap} \cdot D'^2 + 2 \cdot V_{ap} \cdot \hat{d}' \cdot D') \quad (\text{D.24})$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + 2 \cdot K_{SL} \cdot D' \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D'^2 \right) \quad (\text{D.25})$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + 2 \cdot K_{SL} \cdot D'} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D'^2 \right) \right) \quad (\text{D.26})$$

EPCM1

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + V_{SL} \cdot d \quad (D.27)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + V_{SL} \cdot \hat{d} \quad (D.28)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \quad (D.29)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \right) \quad (D.30)$$

EPCM2

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + K_{SL} \cdot v_{ap} \cdot d \quad (D.31)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + K_{SL} \cdot (\hat{v}_{ap} \cdot D + V_{ap} \cdot \hat{d}) \quad (D.32)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \right) \quad (D.33)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \right) \right) \quad (D.34)$$

EVCM1

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} - V_{SL} \cdot d' \quad (D.35)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot \left(\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap} \right) - V_{SL} \cdot \hat{d}' \quad (D.36)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \quad (D.37)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) \right) \quad (D.38)$$

EVCM2

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} - K_{SL} \cdot v_{ap} \cdot d' \quad (D.39)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot \left(\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap} \right) - K_{SL} \cdot \left(\hat{v}_{ap} \cdot D' + V_{ap} \cdot \hat{d}' \right) \quad (D.40)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} \right) + \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D' \right) \quad (D.41)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + K_{SL}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D' \right) \right) \quad (D.42)$$

VCM3

Write the equation for the voltage at v_C :

$$v_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} - K_{SL} \cdot v_{ap} \cdot d' \quad (D.43)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) - K_{SL} \cdot (\hat{v}_{ap} \cdot D' + V_{ap} \cdot \hat{d}') \quad (D.44)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D' \right) \quad (D.45)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D' \right) \right) \quad (D.46)$$

EPCM3

Write the equation for the voltage at v_C :

$$v_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + K_{SL} \cdot d \cdot d' \cdot v_{ap} + V_{SL} \cdot d \quad (D.47)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \left(-0.5 \cdot R_i \cdot \frac{T}{L} + K_{SL} \right) \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + V_{SL} \cdot \hat{d} \quad (D.48)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(\left(-0.5 \cdot R_i \cdot \frac{T}{L} + K_{SL} \right) \cdot (D' - D) + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \cdot D' \right) \quad (D.49)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + (1 - 2 \cdot D) \cdot K_{SL} + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \cdot D' \right) \right) \quad (D.50)$$

EPCM4

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + K_{SL} \cdot v_{ap} \cdot d + V_{SL} \cdot d \quad (\text{D.51})$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + K_{SL} \cdot (\hat{v}_{ap} \cdot D + V_{ap} \cdot \hat{d}) + V_{SL} \cdot \hat{d} \quad (\text{D.52})$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \right) \quad (\text{D.53})$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL} + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \right) \right) \quad (\text{D.54})$$

Appendix E – A More Complete Unified Model

For PCM2, VCM2 and EPCM3 in Table 1, the proportional slope-compensation term K_{SL} appears in the closed current-loop sampling gain K_e . This should not be required, as K_{SL} represents a physical gain block. Reviewing the original modulator derivations, $v_{ap} \cdot d$ can be expressed as v_{cp} , which is consistent with the physical circuit. For each topology, this allows the terminal voltages to be specified as:

$$\text{Buck} \quad \hat{v}_{ap} = \hat{v}_{IN} \quad \hat{v}_{cp} = \hat{v}_O \quad \frac{V_{cp}}{V_{ap}} = \frac{V_O}{V_{IN}} = D \quad (\text{E.1})$$

$$\text{Boost} \quad \hat{v}_{ap} = \hat{v}_O \quad \hat{v}_{cp} = \hat{v}_O - \hat{v}_{IN} \quad \frac{V_{cp}}{V_{ap}} = \frac{V_O - V_{IN}}{V_O} = D \quad (\text{E.2})$$

$$\text{Buck-boost} \quad \hat{v}_{ap} = \hat{v}_{IN} + \hat{v}_O \quad \hat{v}_{cp} = \hat{v}_O \quad \frac{V_{cp}}{V_{ap}} = \frac{V_O}{V_{IN} + V_O} = D \quad (\text{E.3})$$

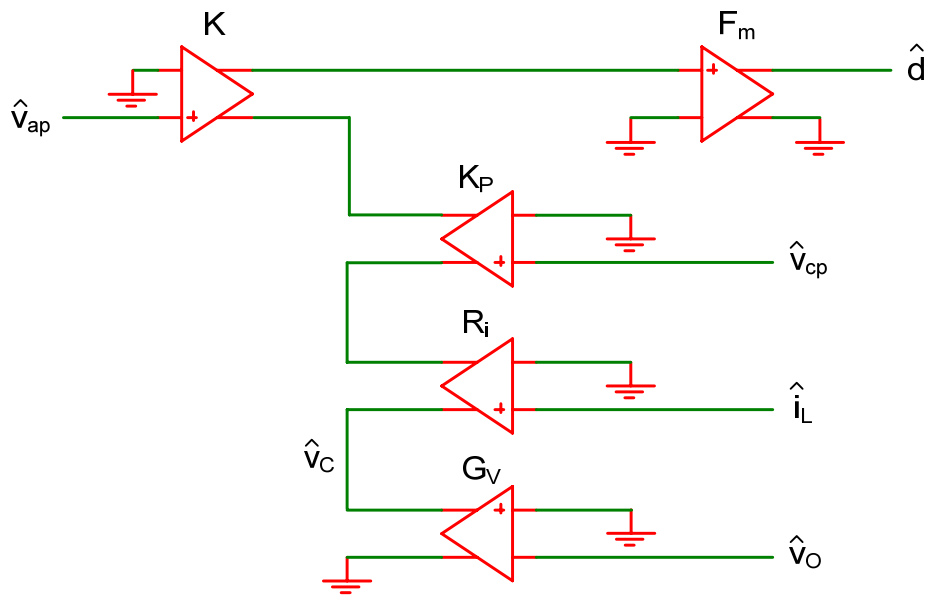


Figure E-1. A more complete unified control-stage model. This accommodates the proportional slope compensation as a feed-forward term using K_P .

For PCM2 and VCM2, this allows the current-loop gain to be accurately modeled using $H_p(s)$.

TABLE E-1 REVISED GENERAL GAIN PARAMETERS				
		$V_{SLOPE} = S_e \cdot T$		$S_{ap} = S_n + S_f$
Mode	S_e, S_n, S_f, S_{ap}	m_c, Q	K_m, K	K_p, K_e
PCM2	$S_e = \frac{V_{ap} \cdot D \cdot K_{SL}}{T}$ $S_n = \frac{V_{ap} \cdot D' \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_n}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D' - 0.5)}$	$K_m = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + K_{SL} \cdot D}$ $K = 0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D'$	$K_p = K_{SL} \cdot D$ $K_e = 0$
VCM2	$S_e = \frac{V_{ap} \cdot D' \cdot K_{SL}}{T}$ $S_f = \frac{V_{ap} \cdot D \cdot R_i}{L}$	$m_c = 1 + \frac{S_e}{S_f}$ $Q = \frac{1}{\pi \cdot (m_c \cdot D - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL} \cdot D'}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D'$	$K_p = K_{SL} \cdot D'$ $K_e = 0$
EPCM3	$S_e = \frac{V_{ap} \cdot D' \cdot K_{SL} + V_{SL}}{T}$ $S_{ap} = \frac{V_{ap} \cdot R_i}{L}$	$m_c = \frac{S_e}{S_{ap}}$ $Q = \frac{1}{\pi \cdot (m_c - 0.5)}$	$K_m = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL} \cdot D' + \frac{V_{SL}}{V_{ap}}}$ $K = -0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D$	$K_p = -K_{SL} \cdot D$ $K_e = -D \cdot T$

PCM2

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + K_{SL} \cdot v_{cp} \cdot d \quad (E.4)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i + 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + K_{SL} \cdot (\hat{v}_{cp} \cdot D + V_{cp} \cdot \hat{d}) \quad (E.5)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} \cdot \frac{V_{cp}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) + \hat{v}_{cp} \cdot (K_{SL} \cdot D) \quad (E.6)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(0.5 - D) \cdot R_i \cdot \frac{T}{L} + K_{SL} \cdot D} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' \right) - \hat{v}_{cp} \cdot (K_{SL} \cdot D) \right) \quad (E.7)$$

VCM2

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} - K_{SL} \cdot (v_{ap} - v_{cp}) \cdot d' \quad (E.8)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) - K_{SL} \cdot (\hat{v}_{ap} - \hat{v}_{cp}) \cdot D' - K_{SL} \cdot (V_{ap} - V_{cp}) \cdot \hat{d}' \quad (E.9)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} \cdot \frac{V_{ap} - V_{cp}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D' \right) + \hat{v}_{cp} \cdot (K_{SL} \cdot D')$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL} \cdot D'} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' - K_{SL} \cdot D' \right) - \hat{v}_{cp} \cdot (K_{SL} \cdot D') \right) \quad (E.11)$$

EPCM3

Write the equation for the voltage at v_C :

$$v_C = i_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot d \cdot d' \cdot v_{ap} + K_{SL} \cdot (v_{ap} - v_{cp}) \cdot d + V_{SL} \cdot d \quad (E.12)$$

Perturb variables:

$$\hat{v}_C = \hat{i}_L \cdot R_i - 0.5 \cdot R_i \cdot \frac{T}{L} \cdot (\hat{d} \cdot D' \cdot V_{ap} + D \cdot \hat{d}' \cdot V_{ap} + D \cdot D' \cdot \hat{v}_{ap}) + K_{SL} \cdot (\hat{v}_{ap} - \hat{v}_{cp}) \cdot D + K_{SL} \cdot (V_{ap} - V_{cp}) \cdot \hat{d} + V_{SL} \cdot \hat{d} \quad (E.13)$$

Regroup and factor terms:

$$\hat{v}_C = \hat{i}_L \cdot R_i + \hat{d} \cdot V_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot (D' - D) + K_{SL} \cdot \frac{V_{ap} - V_{cp}}{V_{ap}} + \frac{V_{SL}}{V_{ap}} \right) + \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \right) + \hat{v}_{cp} \cdot (-K_{SL} \cdot D) \quad (E.14)$$

The duty cycle equation is:

$$\hat{d} \cdot V_{ap} = \frac{1}{(D - 0.5) \cdot R_i \cdot \frac{T}{L} + K_{SL} \cdot D' + \frac{V_{SL}}{V_{ap}}} \cdot \left(\hat{v}_C - \hat{i}_L \cdot R_i - \hat{v}_{ap} \cdot \left(-0.5 \cdot R_i \cdot \frac{T}{L} \cdot D \cdot D' + K_{SL} \cdot D \right) - \hat{v}_{cp} \cdot (-K_{SL} \cdot D) \right) \quad (E.15)$$

Appendix F – Sampling Gain Review

Reference [1] covers the derivation of the closed current-loop sampling-gain term $H(s)$, starting from the continuous-time model of reference [2]. With $Z_L = s \cdot L$ the simplified relationship was shown to be:

$$H(s) = H_e(s) + \frac{s \cdot L}{R_i} \cdot \left(\frac{1}{K'_{mp}} - \frac{1}{K_m} \right) \quad (F.1)$$

Where $H_e(s) = 1 - \frac{s \cdot T}{2} + \frac{s^2}{\omega_n^2}$ from the continuous-time model, and $\omega_n = \frac{\pi}{T}$. This allows $H(s)$ to be written as:

$$H(s) = 1 + s \cdot K_e + \frac{s^2}{\omega_n^2} \quad \text{where} \quad K_e = \frac{L}{R_i} \cdot \left(\frac{1}{K'_{mp}} - \frac{1}{K_m} \right) - \frac{T}{2} \quad (F.2)$$

K'_{mp} is the ideal (unperturbed) steady-state modulator voltage gain. In general terms:

$$K'_{mp} = \frac{V_{ap}}{(S_{sns} + S_e) \cdot T} \quad (F.3)$$

Where S_{sns} is the naturally-sampled (sensed) inductor slope and S_e is the external (slope-compensating) ramp.

Given the results from Appendix E, much insight into the physical interpretation of the sampling-gain term is made. The term $\frac{s^2}{\omega_n^2}$ shows that a 180° phase shift occurs at half the switching frequency. No useful signal from the control voltage will be accurately reproduced above this frequency. For the emulated architectures, K_e represents the time delay (or phase shift) due to the sample-and-hold function. For the naturally-sampled peak and valley modes, $K_e = 0$.

This relationship for the naturally-sampled modes was first derived in reference [3], where $H(s)$ reduced to $1 + \frac{s^2}{\omega_n^2}$.

In reference [1], the general term for Q was derived as:

$$Q = \frac{1}{\pi \cdot \left(\frac{L}{K'_{mp} \cdot R_i \cdot T} - 0.5 \right)} \quad (F.4)$$

For the naturally-sampled peak and valley modes, Q reduces to $\frac{\omega_L}{\omega_n}$, where $\omega_L = \frac{K_m \cdot R_i}{L}$. This allows the forward-path sampling-gain term for these modes to be expressed as:

$$H_p(s) = \frac{1}{1 + s \cdot \frac{Q}{\omega_n}} \quad \text{or} \quad H_p(s) = \frac{1}{1 + s \cdot \frac{\omega_L}{\omega_n^2}} \quad (\text{F.5})$$

This was first derived in reference [3], and independently arrived at in reference [1].

To date, there are several limiting factors on the use of the sampling-gain terms. The sampling-gain terms are only valid with respect to the modulator voltage gain K_m . The derivations do not include the feed-forward terms K and K_p . Any transfer function which includes the feed-forward terms may or may not be accurate to half the switching frequency, depending on the relative weight and placement of these terms.

To accurately model the current loop, the forward-path sampling-gain term must be used. For the emulated (sample-and-hold) modes, K_e is not accounted for in $H_p(s)$. Consequently there is not a correct linear solution for the emulated mode current loop.

Derivation of $f_L(Q)$

The effective sampled-gain inductor pole $f_L(Q)$ is defined as the frequency at which a 45° phase shift occurs due to the sampling gain. This identifies an upper frequency limit for the voltage loop crossover, while still maintaining adequate phase margin.

For a second-order system such as an LC filter, the frequency response is given by:

$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{s}{\omega_0 \cdot Q} + \frac{s^2}{\omega_0^2}} \quad (\text{F.6})$$

Where ω_0 is the resonant frequency and Q is the quality factor. The magnitude is given by:

$$\left| \frac{v_2}{v_1} \right| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2} \cdot \left(\frac{\omega}{\omega_0}\right)^2}} \quad (\text{F.7})$$

The phase is given by:

$$\angle \frac{v_2}{v_1} = -\tan^{-1} \left(\frac{\frac{1}{Q} \cdot \left(\frac{\omega}{\omega_0}\right)}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \right) \quad (\text{F.8})$$

For the effective sampled-gain inductor pole, let:

$$\omega = 2 \cdot \pi \cdot f_L(Q) \quad \omega_0 = \omega_n = \frac{\pi}{T} \quad (\text{F.9})$$

Combining equations:

$$\frac{\omega}{\omega_0} = 2 \cdot T \cdot f_L(Q) \quad (\text{F.10})$$

Since $f_L(Q)$ is defined as the frequency at which the phase shift is 45° :

$$\tan \angle \frac{v_2}{v_1} = \tan(-45^\circ) = -1 \quad (\text{F.11})$$

Combining equations F.8, F.10 and F.11:

$$1 = \frac{\frac{1}{Q} \cdot 2 \cdot T \cdot f_L(Q)}{1 - (2 \cdot T \cdot f_L(Q))^2} \quad (\text{F.12})$$

Rearranging equation F.12 into quadratic form:

$$4 \cdot T^2 \cdot Q \cdot (f_L(Q))^2 + 2 \cdot T \cdot f_L(Q) - Q = 0 \quad (\text{F.13})$$

$$x = f_L(Q) \quad a = 4 \cdot T^2 \cdot Q \quad b = 2 \cdot T \quad c = -Q$$

The standard quadratic roots are given by:

$$x = \frac{b}{2 \cdot a} \cdot \left(-1 \pm \sqrt{1 - \frac{4 \cdot a \cdot c}{b^2}} \right) \quad (\text{F.14})$$

Substitution of variables for the positive root leads to:

$$f_L(Q) = \frac{1}{4 \cdot T \cdot Q} \cdot \left(\sqrt{1 + 4 \cdot Q^2} - 1 \right) \quad (\text{F.15})$$

An alternate form is found using an improved formula for quadratic roots, which has better accuracy when $4 \cdot Q^2 \ll 1$. The roots are given by:

$$x_1 = -\frac{c}{b} \cdot \frac{1}{F} \quad x_2 = -\frac{b}{a} \cdot F \quad \text{where} \quad F = \frac{1}{2} \cdot \left(1 + \sqrt{1 - \frac{4 \cdot a \cdot c}{b^2}} \right) \quad (\text{F.16})$$

Substitution of variables into x_1 for the positive root leads to:

$$f_L(Q) = \frac{Q}{T} \cdot \left(\frac{1}{1 + \sqrt{1 + 4 \cdot Q^2}} \right) \quad (\text{F.17})$$

Reference:

Dr. R. David Middlebrook, "New Structured Analog Design," Course Revision 3
 Chapter 4. An Improved Formula for Quadratic Roots
 Chapter 5. Approximations and Assumptions